Concentration bounds for RIC^2 (Randomized Incremental Construction)

Sandeep Sen

IIT Delhi, India

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[2]to appear in STACS'19

Once upon a time ..

Conceived : 1986 *Forgotten: 1993*

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Randomized Incremental Construction (RIC)

Starting from an empty set

Repeat:

1. *Insert the next object*
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Expected total time $= \max_{input\ I} \mathbb{E}[T_s(I)]$
(worst case for any input of size n).

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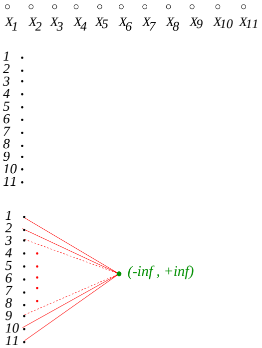
Quicksort as R.I.C.

Gradual refinement of partition



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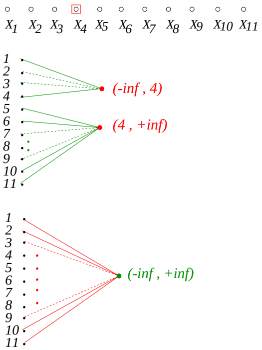
Conflict graph



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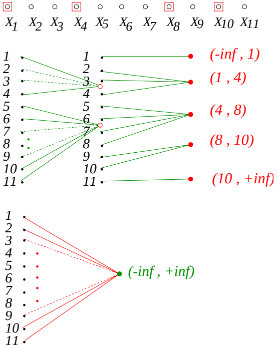

Bounding the maximum sub-problem size

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σ: a subproblem is defined by $D(\sigma)$ elements
$I(\sigma)$: size of the subproblem (unchosen elements in σ)

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$\Pi(n) = \binom{n}{2}$ pairs of points

A subproblem is defined by a pair of sample points
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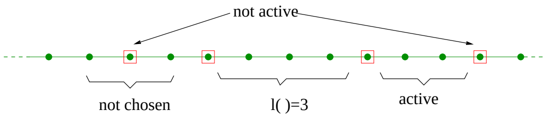

$\Pi^0(n)$ special significance : $l(\sigma) = 0$

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Bounding the maximum sub-problem

Claim :

$$\Pr\{\max_{active\ \sigma} l(\sigma) \geq c\frac{n}{r}\log r\} \leq \frac{1}{2}$$

R chosen by Bernoulli sampling $p = r/n$

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$p(\sigma, r)$: conditional probability that none of the k $(= l(\sigma))$ elements are selected given $D(\sigma)$ chosen

$\leq (1 - r/n)^k$

$\leq e^{-c\log r} = 1/r^c$ for $\boxed{k \geq cn/r \ln r}$

BAD σ

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$$\leq \frac{1}{r^c} \sum_{\sigma \in \Pi(n)} \Pr[D(\sigma) \subset R] = \frac{1}{r^c} \sum_{\sigma \in \Pi(n)} E[D(\sigma) \subset R]$$

$$= \frac{1}{r^c} \quad E[\text{ number sub-problems for which } D(\sigma) \subset R]$$

(linearity of Expectation)

$= r^{O(1)}$ for $D(\sigma) = O(1)$

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$\leq 1/2$ for appropriate c

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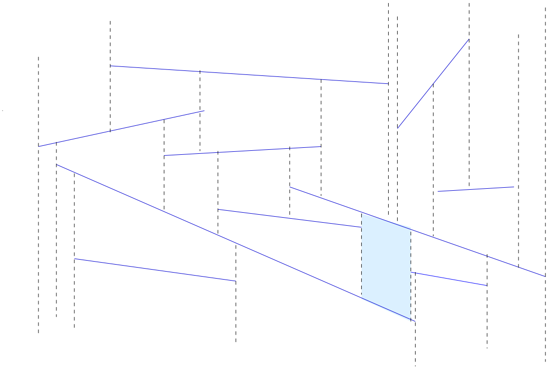
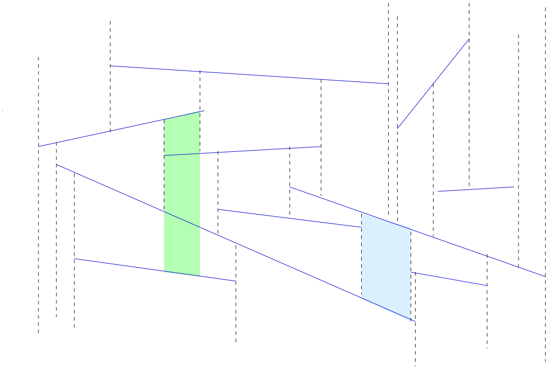
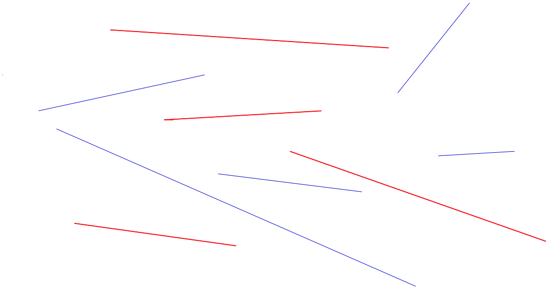
More general : Trapezoidal Map



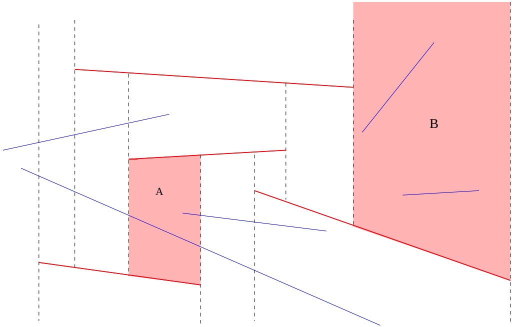
Trapezoidal Map : Ranges



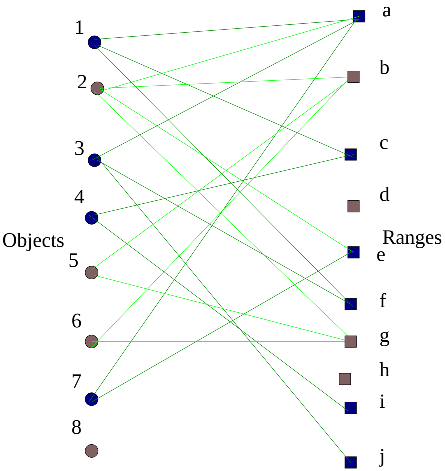
Randomized Incremental Construction



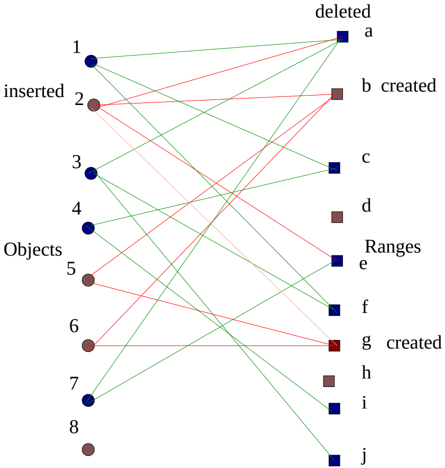
Objects (segments) and ranges (trapezoids)



A more general scenario



Modifications caused by insertion of an object



A general bound for RIC [Cl-Sh]

Total (amortised) cost = O(edges created in conflict graph)

Edges can be deleted at most once

General Step: $R \leftarrow R \cup s$ (both random subsets)

Expected work (#edges created in the conflict graph)=

$$\sum_{\sigma \in \Pi^0(R \cup s)} l(\sigma) \cdot \Pr\{\sigma \in \Pi^0(R \cup s) - \Pi^0(R)\}$$

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From *backward analysis* this probability is the same as deleting a random element from $R \cup s$ which is $\frac{d(\sigma)}{r+1}$. Substituting

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Sum bound

$$\sum_{\sigma \in \Pi^0(R \cup s)} l(\sigma) = \frac{n}{r} \cdot E[\Pi^0(R \cup s)]$$

[23] to appear in STACS'19

A general bound on Expected running time of RIC

$$= O(\frac{d(\sigma)}{r} \cdot \frac{n}{r} E[\Pi^0(R \cup s)])$$

A common scenario $E[\Pi^0(R) = O(r)$.

$$\text{Total expected cost of RIC } = \sum_{r=1}^{r=n} O\left(\frac{d}{r} \cdot n\right)$$

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OPEN PROBLEM

Tail estimates in the general case

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Tail estimates in the general case without independent repetitions

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Sum of subproblem sizes

Def: c-order conflict $\binom{l(\sigma)}{c}$, for some $c \geq 0$

Let $T_c = \sum_{\sigma \in \Pi^0(R)} \binom{l(\sigma)}{c}$

Remark For technical reasons it is not $l(\sigma)^c$. $T_0 = |\Pi^0(R)|$. $T_1 =$ sum of subproblems.

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$$\textbf{Claim } E[T_c] = O(\left(\tfrac{n}{r}\right)^c E[\Pi^c(R)])$$

For constant c, $E[\Pi^c(R) = O(E[\Pi^0(R)]$ implying that average conflict size is very close to $\frac{n}{r}$

[27] to appear in STACS'19

Sum of subproblem sizes

$T_c = \sum_{\sigma \in \Pi(N)(R)} \binom{l(\sigma)}{c} I_{\sigma,R}$ where $I_{\sigma,R} = 1$ if $\sigma \in \Pi^0(R)$.

$E[T_c] = \sum_{\sigma \in \Pi(N)} \binom{l(\sigma)}{c} p^{d(\sigma)} \cdot (1-p)^{l(\sigma)}$ for $l(\sigma) \geq c$.

$= \sum_{\sigma \in \Pi(N)} \binom{l(\sigma)}{c} p^{d(\sigma)+c} \cdot (1-p)^{l(\sigma)-c} \cdot \left(\frac{1-p}{p}\right)^c$

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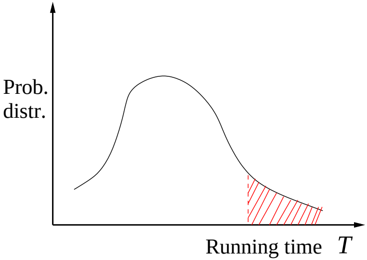
Standard Tools

Running time is bounded by some probability distribution



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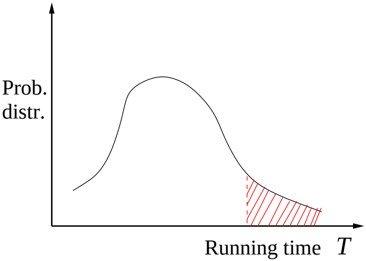

Probabilistic Inequalities :

- Markov $\Pr[X > kE[X]] < 1/k$
- Chernoff $\Pr[X > A] \leq G_X(s) \cdot s^{-A}$
- Chebychev $\Pr[|X - E[X]| > r] \leq \sigma^2/r^2$

Notation : $\widetilde{O}(\cdot) \stackrel{def}{=} O(\cdot)$ with prob. $1 - \frac{1}{n}$ Inv polynomial

[31] to appear in STACS'19

A martingale framework

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Deviation from Expectation

Z_0 : expected runnning time of ric and Z_n: actual runnning time
Concentration bound : $\Pr[|Z_0 - Z_n| \geq \lambda] < ???$

[33] to appear in STACS'19

A martingale inequality

Theorem [Freedman 75]

Let $X_1, X_2 \ldots X_n$ be a sequence of random variables and let Y_k, a function of $X_1 \ldots X_k$ be a martingale sequence, i.e., $\mathbb{E}[Y_k | X_1 \ldots X_k] = Y_{k-1}$ such that $\max_{1 \leq k \leq n}\{|Y_k - Y_{k-1}|\} \leq M_n$. Let

$$W_k = \sum_{j=1}^{k} \mathbb{E}[(Y_j - Y_{j-1})^2 | X_1 \ldots X_{j-1}] = \sum_{j=1}^{k} Var(Y_j | X_1 \ldots X_{j-1})$$

where *Var* is the variance using $\mathbb{E}[Y_j] = Y_{j-1}$. Then for all λ and $W_n \leq \Delta^2, \ \ \Delta^2 > 0$,

$$\Pr[|Y_n - Y_0| \geq \lambda] \leq 2\exp\left(-\frac{\lambda^2}{2(\Delta^2 + M_n \cdot \lambda/3)}\right)$$

Martingale inequality

Extended Freedman inequality

Let $\Pr[\max_{1 \le k \le n}\{|Y_k - Y_{k-1}|\} \ge M_n, W_n \ge \Delta^2] \le \frac{1}{f(n)}$, then,

$$\Pr[|Y_n - Y_0| \ge \lambda] \le 2\exp\left(-\frac{\lambda^2}{2(\Delta^2 + M_n \cdot \lambda/3)}\right) + O(1/f(n))$$

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Azuma-Hoeffding

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[36] to appear in STACS'19

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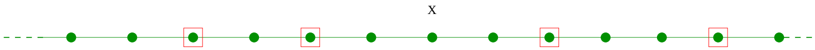
Mehlhorn-Sharir-Welzl [93] obtained a special kind of Martingale concentration bound that was effective for some cases of line segment intersection RIC.

[36] to appear in STACS'19

Application to quicksort



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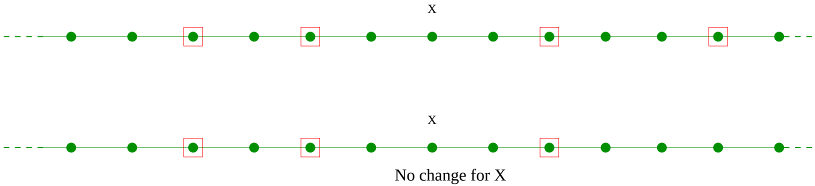

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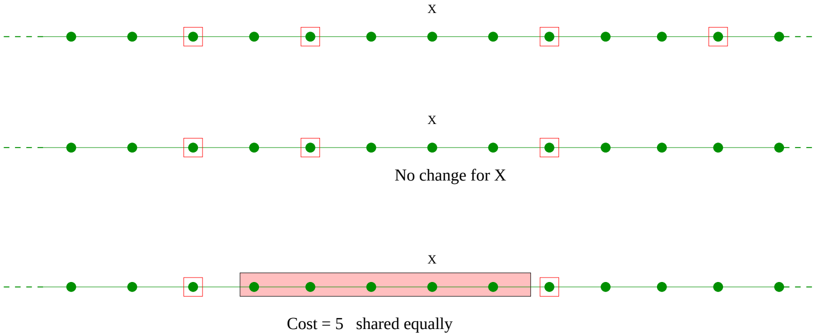

$$I_j^x = \begin{cases} 1 & \text{if interval containing } x \text{ changes in step } j \\ 0 & \text{otherwise.} \end{cases}$$

[38] to appear in STACS'19

Analysis

$$\mathbb{E}[\sum_j I_j^x] = \sum_j \Pr[I_j^x = 1] = \sum_j \frac{2}{j} = \log n$$

Analysis

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$$\mathbb{E}_j[(I_j - E[I_j])^2] = \mathbb{E}_j[I_j^2] - \mathbb{E}_j^2[I_j] \ \le \mathbb{E}_j[I_j^2] - \frac{4}{(n-j)^2}$$

$$\le \frac{2}{n-j} \quad I_j^2 \text{ also 0-1 indicator rv}$$

$$\sum_{j=1}^{n} \mathbb{E}_{j-1}[(Y_j - Y_{j-1})^2] \le \sum_{j=1}^{n} \frac{2}{n-j} \le 2\log n = W_n$$

Inverse polynomial bound for quicksort

$$\Pr[|Y_n - Y_0| \geq c \log n] \leq \exp\left(-\frac{4c^2 \log^2 n}{2(\log n + c \log n/3)}\right) \leq \frac{1}{n^c}.$$

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[42] to appear in STACS'19

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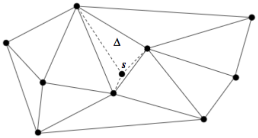
Azuma-Hoeffding

$$\Pr[|Y_n - Y_0| \geq t] \leq \exp\left(\frac{-t^2}{\sum_{i=1}^{n} M_i^2}\right)$$

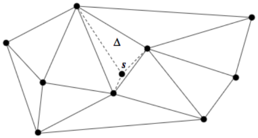
For $M_i = 1$, there is no meaningful bound for this setting.

[42] to appear in STACS'19

Incremental Delaunay triangulation



Incremental Delaunay triangulation



Total size is $O(i)$ but the degree of a vertex can be large

[44] to appear in STACS'19

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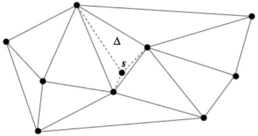

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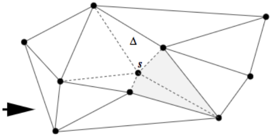

Number of new Δ's is 5 and can be as large as i but ..

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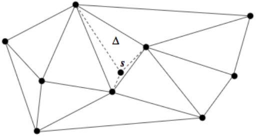

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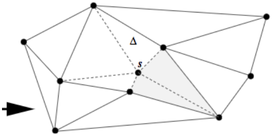

Number of new Δ's is 5 and can be as large as i but ..
the average degree of a planar graph is $O(1)$.

~~Expected number of triangles that appear~~ over the course of RIC $= O(n)$

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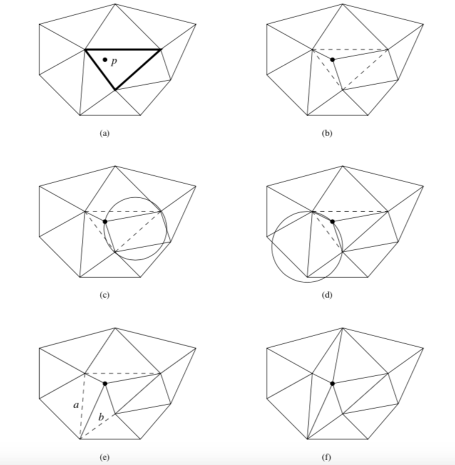
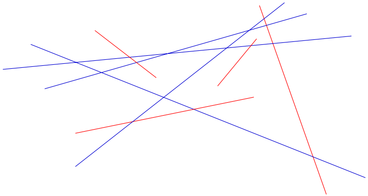
Flip algorithm [Green and Sibson]



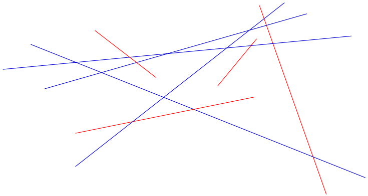
Courtesy: Lischinski[93]

Line segment intersections



No intersections between red. All blue segments intersect.

Line segment intersections



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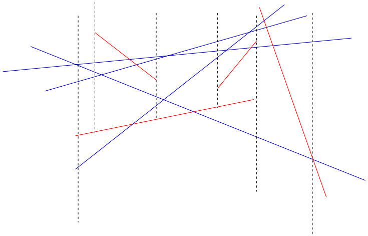

Intermediate structures (trapezoidal maps) can have huge variance

[47] ... in STACS'10

Expected number of intersections at stage i

Probability intersection (s_j, s_k) appear in stage i = Probabilty s_j, s_k have been chosen.

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$$\mathbb{E}[M_i] = m \cdot \frac{i}{n} \cdot \frac{i}{n} = \frac{m \cdot i^2}{n^2}$$

[49] to appear in STACS'19

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Ideal bound for segment intersections : $O(m + n \log n)$

[49] to appear in STACS'19

Summary of results

Problem	Exp Run time	Tail estimates
Quicksort	$O(n \log n)$	$O(c\gamma n \log n)$ w.p. $\geq 1 - n^{-c}$
Delaunay Triangulation	$O(n \log n)$	$O(c\gamma n \log n)$ w.p. $\geq 1 - 2^{-c}$
Segment intersections/ Trapezoidal maps	$O(n \log n + m)$ *no conflict list*	$O(n \log n + m)$ w.p. $1 - \exp -(\frac{m+n \log n}{n\alpha(n)})$
	*using conflict list**	w.p. $\geq 1 - \exp -(\frac{m}{n \log^2 n})$

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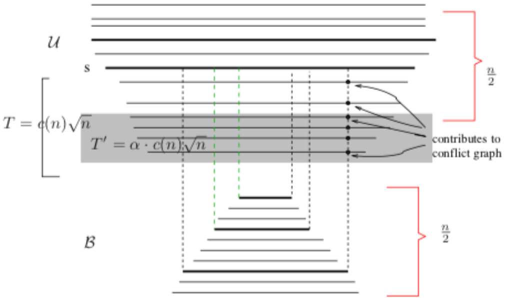
The non-conflict bound is nearly inverse polynomial for $m = 0$, i.e., $\exp -(\log n/\alpha(n))$

[51] to appear in STACS'19

A lower bound construction



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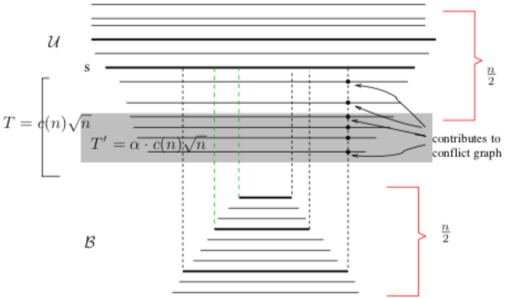

$|\mathcal{U}| = |\mathcal{B}| = \frac{n}{2}$
Among first $3\sqrt{n}$ insertions, $\Pr[\mathcal{B} \geq \sqrt{n}] \geq 1 - 2^{\sqrt{n}}$.

[53] to appear in STACS'19

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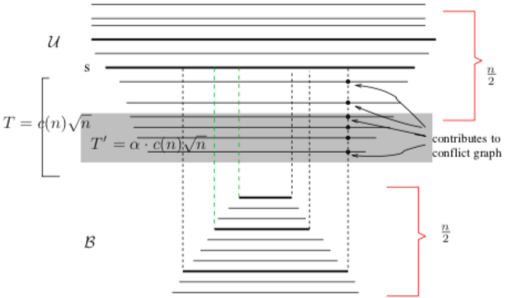

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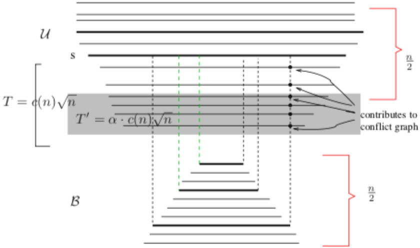
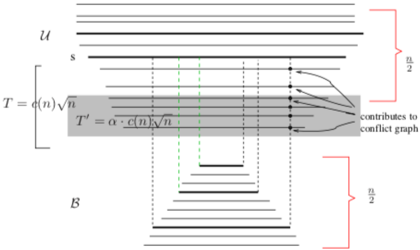
$$\Pr[|T| \geq c(n)\sqrt{n}] \geq 4^{-3c(n)}$$

[53] to appear in STACS'10

Construction cont.



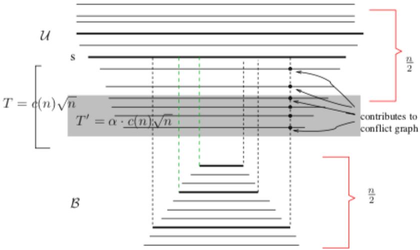
Construction cont.



Number of times s moves down without touching T' (shaded region) $= r$
$\Rightarrow$ RIC incurs $r \cdot c(n)\sqrt{(n)}\sqrt{(n)}$ cost

[55] to appear in STACS'19

Construction cont.



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[55] to appear in STACS'19

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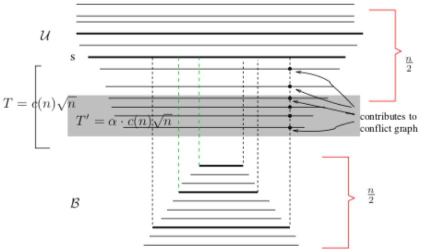

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$$\Pr[X \geq n \log n \log \log n] \geq \frac{1}{\sqrt{n}} \quad \text{for } c(n) = \Omega(\log n)$$

[55] to appear in STACS'19

Future directions

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- Inverse polynomial bounds open for Delaunay, Line segments and many others

[57] to appear in STACS'19

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[57] to appear in STACS'19

Future directions

- Inverse polynomial bounds open for Delaunay, Line segments and many others
- More lower bound constructions
- Distinct variations of RICs (rebuild ?) may have different performance including data structures like conflict lists (without).

[57] to appear in STACS'19