Generalized Matroid Secretary Problem

Sourav Chakraborty
(Indian Statistical Institute)

The Art of Selling a Car

Want to sell my car.

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2 L

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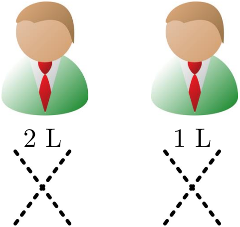

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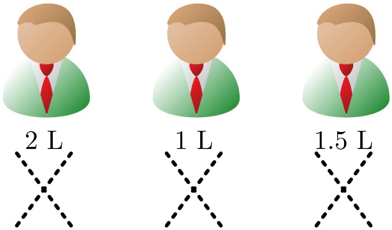

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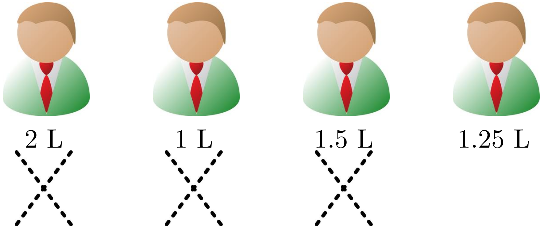

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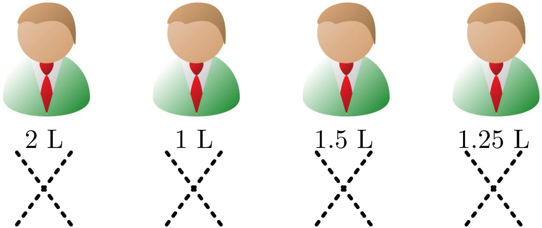

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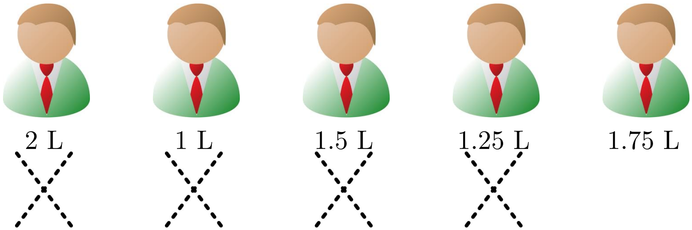

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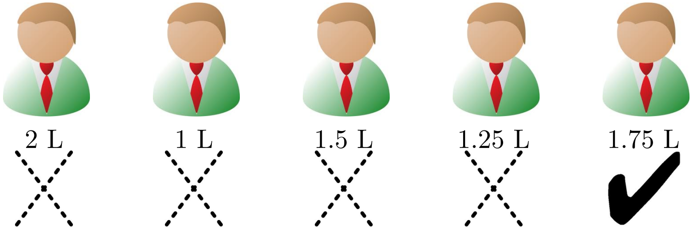

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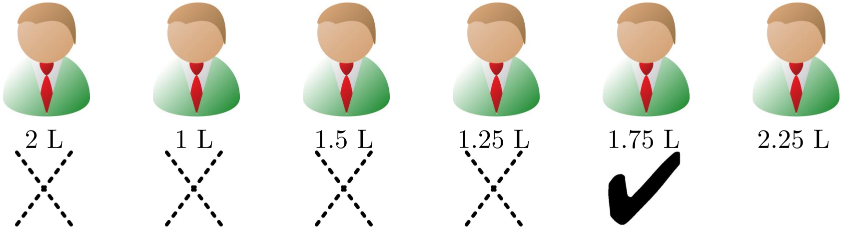

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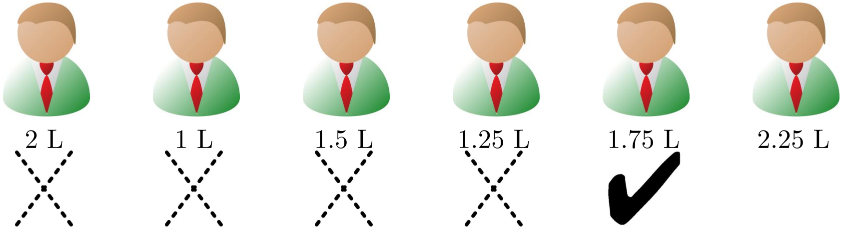

What should be my strategy?

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GOAL: To maximize $\frac{\text{Expected Return}}{\max_i B_i}$.

Algorithm for Secretary Problem

Simple Algorithm

REJECT the bids $B_1, B_2, \ldots, B_{N/2}$
Let $C = \max\{B_1, \ldots, B_{N/2}\}$.
For any $i > N/2$ if B_i is at least C then ACCEPT B_i.

With probability $1/4$ the highest bid is the second half and the second highest bid is in the first half. So,

$$\frac{\text{Expected Return}}{\max_i B_i} > \frac{1}{4}.$$

[Lindley, Dynkin (1963)] showed that the competitive ratio is $1/e$.

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Bidders

B1 Chennai-London-Mumbai ($300)

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B1 Chennai-London-Mumbai ($300) NO

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Bidders

B1 Chennai-London-Mumbai ($300) NO

B2 London-Paris ($100)

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Bidders

B1 Chennai-London-Mumbai (\$300) NO

B2 London-Paris (\$100) YES

B3 Mumbai-Paris-NY (\$700)

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B3 Mumbai-Paris-NY ($700) YES

B4 Mumbai-Paris-NY ($800)

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B3 Mumbai-Paris-NY (\$700) YES

B4 Mumbai-Paris-NY (\$800) YES

B5 Chennai-London-NY (\$950)

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B3 Mumbai-Paris-NY ($700) YES
B4 Mumbai-Paris-NY ($800) YES
B5 Chennai-London-NY ($950) YES
B6 London-Paris-NY ($1200)
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What should be my strategy?

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$(\mathcal{U}, \mathcal{I})$ is a matroid. $\mathcal{U}$ is a universe (all possible itineraries). $\mathcal{I}$ is the set of independent sets (allowed combinations of the elements of the universe).

$$\text{Bids: } (U_1, B_1), (U_2, B_2), \ldots, (U_N, B_N) \in (\mathcal{U}, \mathbb{R}^+)$$

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Theorem (Chakraborty-Lachish 2012)

For the general matroid secretary problem there is a strategy such that the expected return is at least $1/\sqrt{\log d}$ of the OPT.

Current Status

Improvement of the Competitive Ratio

- $O(\log d)$ by Babaioff-Immorlica-Kempe-Klienberg 2008
- $O(\sqrt{\log d})$ by Chakraborty-Lachish 2012
- $O(\log \log d)$ by Lachish 2014
- $O(\log \log d)$ by Moran-Svensson-Zenklusen 2015

Better (even constant competitive ratio) algrotihms are known for special matroids.

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Sampling Phase : Reject $(v_1, w(v_1)), (v_2, w(v_2)), \ldots, (v_{N/2}, w(v_{N/2}))$, but record all the data.

Selection Phase : Based on the data from the Sampling Phase decide which of the $(v_{N/2+1}, w(v_{N/2+1})), \ldots, (v_N, w(v_N))$ to select.

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What is the OPT?

A1 Let $M < OPT/\sqrt{\log d}$ be the maximum weight of all the vectors. M is known to us.

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- A3' Let R_i be the set of vectors in $(v_{N/2+1}, w(v_{N/2+1})), \ldots, (v_N, w(v_N))$ that have weight $OPT/2^i$.

What is the OPT?

- A1 Let $M < OPT/\sqrt{\log d}$ be the maximum weight of all the vectors. M is known to us.
- A2 For all vectors v_i the weight $w(v_i)$ is one of $\{OPT, \frac{OPT}{2}, \frac{OPT}{4}, \ldots, \frac{OPT}{2^{\log d}}\}$.
- A3 Let L_i be the set of vectors in $(v_1, w(v_1)), \ldots, (v_{N/2}, w(v_{N/2}))$ that have weight $OPT/2^i$.
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For all i the best offline algo on the first half will choose $d_i \triangleq dim(L_1 \cup L_2 \cup \cdots \cup L_i) - dim(L_1 \cup L_2 \cup \cdots \cup L_{i-1})$ number of vectors from layer L_i.

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Let us only plan to SELECT elements from L_i and L_j.

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So we have to understand how the layers are disrupting each other.