Parameterized Algorithms for Longest paths and cycles Above Some Natural Lower Bounds

Saket Saurabh

The Institute of Mathematical Sciences, India

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The Longest Path and Cycle problems

Longest Path

Input: A graph G and a positive integer k.

Task: Decide whether G contains a path with (at least) k vertices.

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Some History (Longest Path)

Reference	Randomized	Deterministic
Monien85	-	$\mathcal{O}(k!nm)$
Bodlaender	-	$\mathcal{O}(k!2^k n)$
Alon, Y and Z	$\mathcal{O}(5.44^k n)$	$\mathcal{O}(c^k n \log n)$ for a large c
Huffner, W, and Z	$\mathcal{O}(4.32^k m)$	
Kneis, M, R, and R	$\mathcal{O}^*(4^k)$	$\mathcal{O}^*(16^k)$
Chen, L, S, and Z	$\mathcal{O}(4^k k^{2.7} m)$	$4^{k+\mathcal{O}(\log^3 k)} nm$
Koutis	$\mathcal{O}^*(2.83^k)$	-
Williams	$\mathcal{O}^*(2^k)$	-
Bjorklund, H, K, and K	$\mathcal{O}^*(1.66^k)$	-
Fomin, L, and S	-	$\mathcal{O}(2.851^k n \log^2 n)$
F, L, P, and S	-	$\mathcal{O}(2.619^k n \log n)$
Zehavi		$\mathcal{O}^*(2.5961^k)$

Puzzle

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1. Assume that you have an oracle $\mathcal{A}$ that can test whether there is a cycle of length ℓ in an undirected graph G in time $\mathcal{O}^*(2^\ell)$.
2. Can you use $\mathcal{A}$ to solve Longest Cycle in time $\mathcal{O}^*(2^{\mathcal{O}(k)})$.

The Longest Path and Cycle problems

Theorem (Zehavi, 2015, 2017)

Longest Path and Longest Cycle can be solved in times $2.59606^k \cdot n^{\mathcal{O}(1)}$ and $4^k \cdot n^{\mathcal{O}(1)}$ (randomized) respectively.

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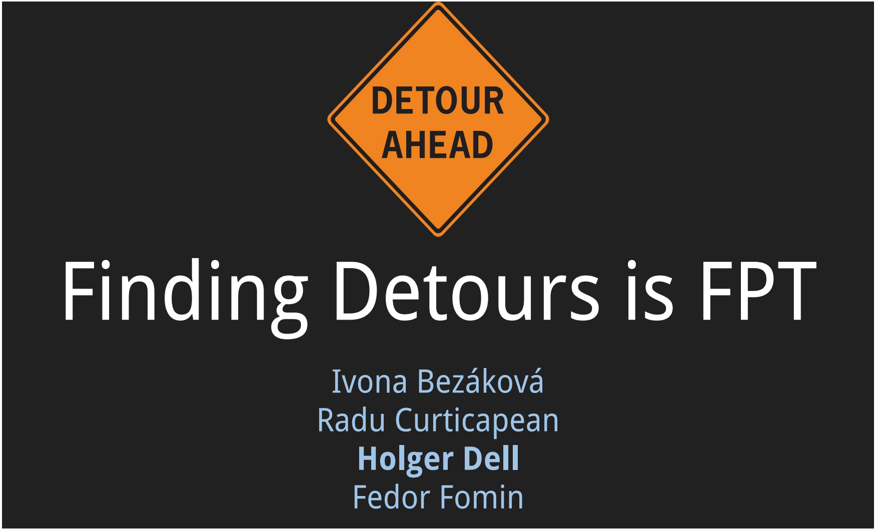
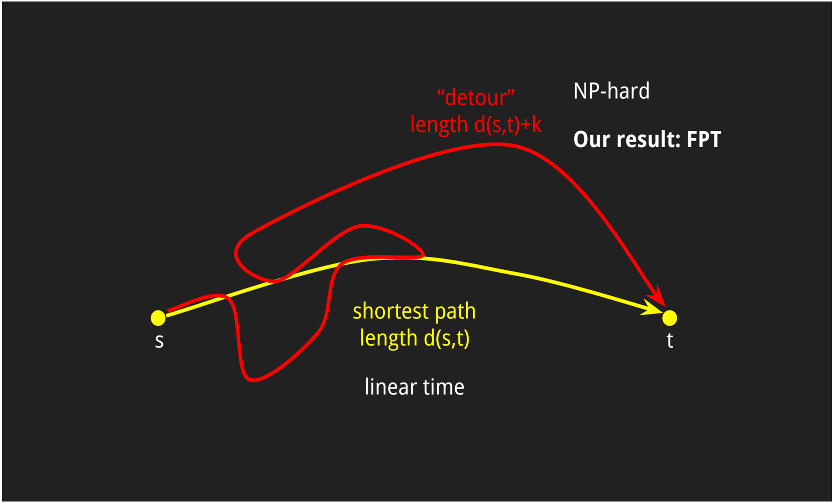
Theorem (Zehavi, 2015, 2017)

Longest Path and Longest Cycle can be solved in times $2.59606^k \cdot n^{\mathcal{O}(1)}$ and $4^k \cdot n^{\mathcal{O}(1)}$ (randomized) respectively.

- Longest Cycle can be solved deterministically in time $4.884^k \cdot n^{\mathcal{O}(1)}$ respectively.

Above Guarantee Parameterization





"detour"
length d(s,t)+k
NP-hard
Our result: FPT
shortest path
length d(s,t)
linear time
s
t

Is there a path from s to t of length...

≤ k ? → Shortest Path

≥ k ? → Longest Path

= k ? → Exact Path

randomized time $O^*(1.657^k)$
[Björklund, Husfeldt, Kaski, Koivisto 2010]

deterministic time $O^*(2.597^k)$
[Zehavi 2015]

Observation: Algorithms bad when $k < d(s,t) \sim n^{0.1}$

Detour = "Above guarantee" Longest Path

Is there a path from s to t of length...

$\geq d(s,t) + k$? → Detour

$= d(s,t) + k$? → Exact Detour

Our result: Both variants are FPT

Actual talk will be:

Longest Path and Cycle Above Degeneracy

Degeneracy and paths (cycles)

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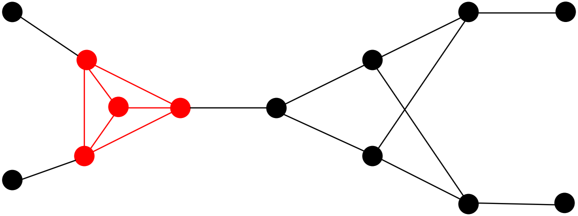

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Proposition

A graph G of degeneracy d contains a path with at least $d+1$ vertices. If $d \geq 2$, then G contains a cycle with at least $d+1$ vertices.

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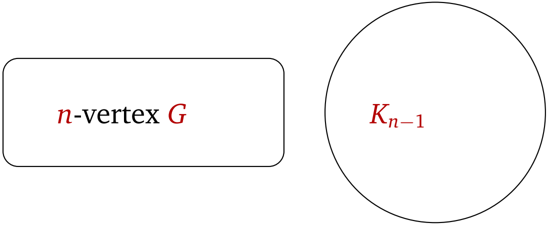

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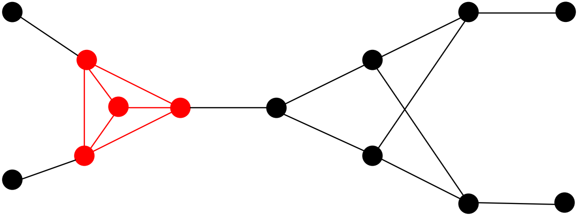

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Let $\{s_1, t_1\}, \ldots, \{s_r, t_r\}$, $r \leq k$, be a collection of pairs of vertices of H such that

- $s_i \neq t_j$ for all $i \neq j$, $i, j \in \{1, \ldots, r\}$,
- $s_i \neq s_j$ for all $i \neq j$, $i, j \in \{1, \ldots, r\}$, and
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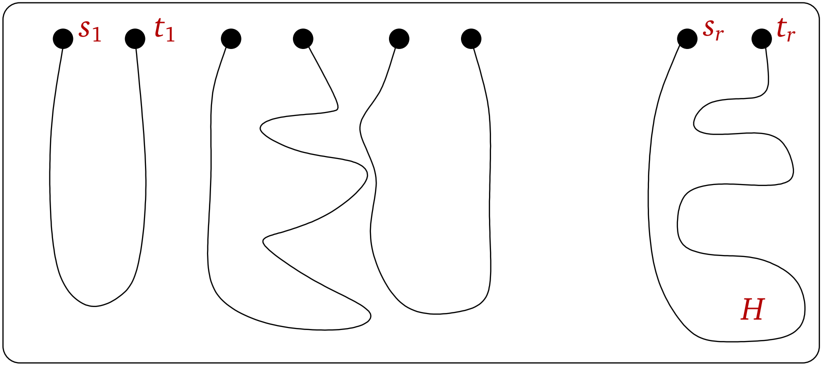
- there is at least one index $i \in \{1, \ldots, r\}$ such that $s_i \neq t_i$.

Claim: There is a family of pairwise vertex-disjoint paths $\mathcal{P} = \{P_1, \ldots, P_r\}$ in H such that each P_i is an (s_i, t_i)-path and $\cup_{i=1}^{r} V(P_i) = V(H)$, that is, the paths cover all vertices of H.

Cores and paths



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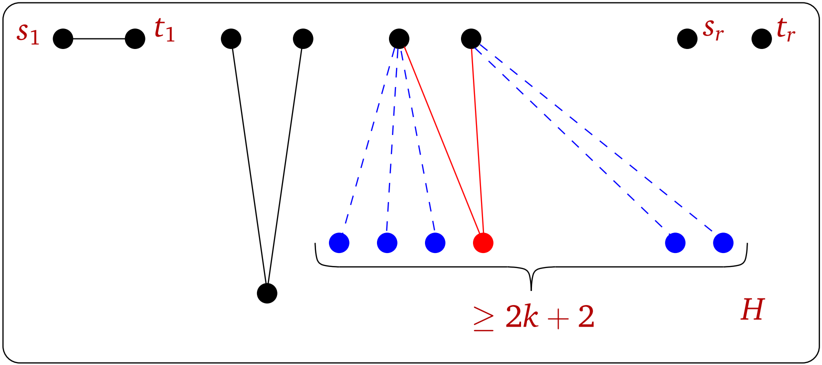

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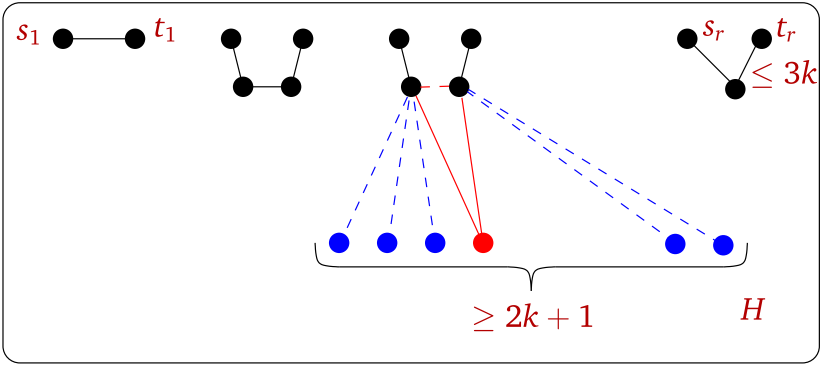

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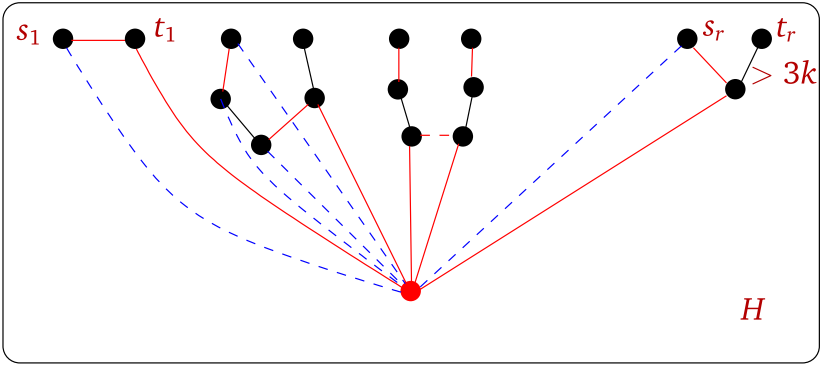

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Claim: If G has a path with $d + k$ vertices, then G has a path P with $d + k$ vertices such that $V(H) \subseteq V(P)$ and at least one end-vertex of P is outside H.

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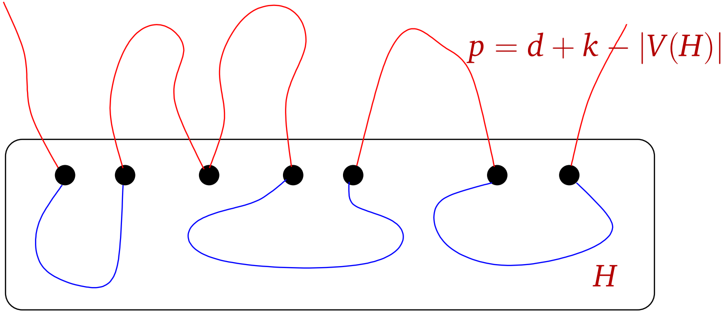

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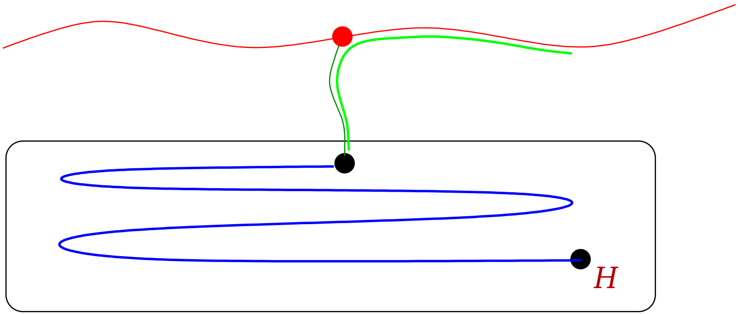

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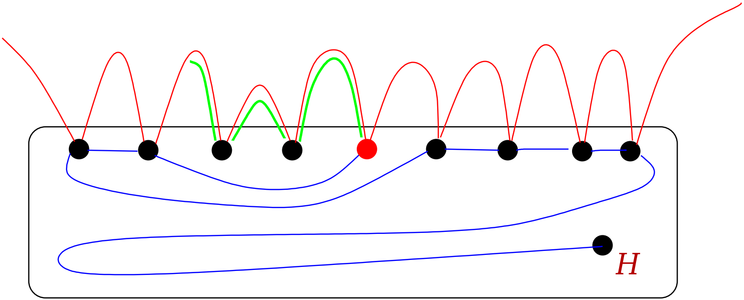

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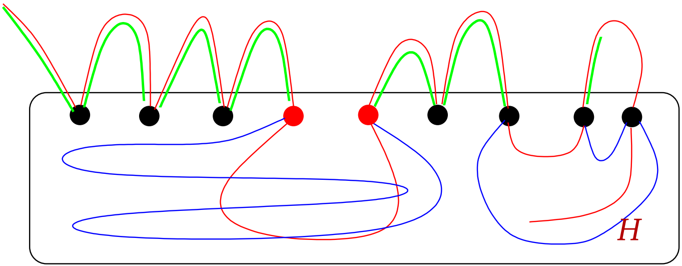

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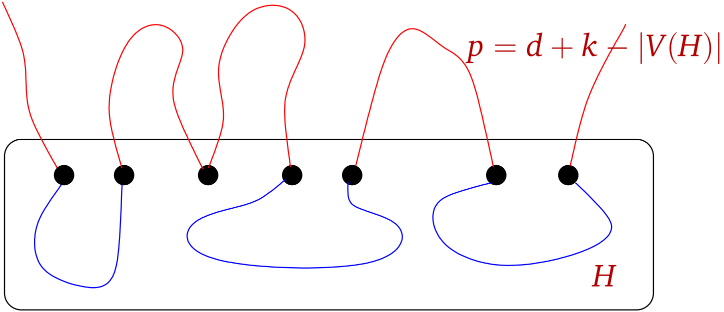

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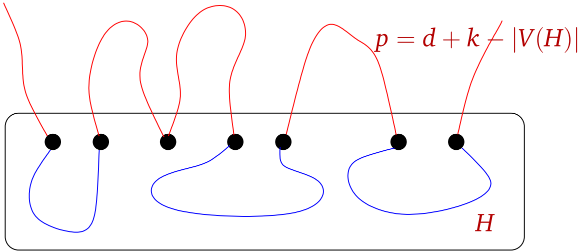
Claim: If G has a family of internally vertex disjoint paths s.t.

- there are at least one and at most two paths that have its one end-vertex outside $V(H)$ and the second in $V(H)$, and the other paths have their end-vertices in $V(H)$,
- the union of the paths is a linear forest such that if two paths have end-vertices outside $V(H)$, then they are in distinct component of the forest,
- the total number of internal vertices is $p = d + k - |V(H)|$,

then G has a path with $d + k$ vertices.



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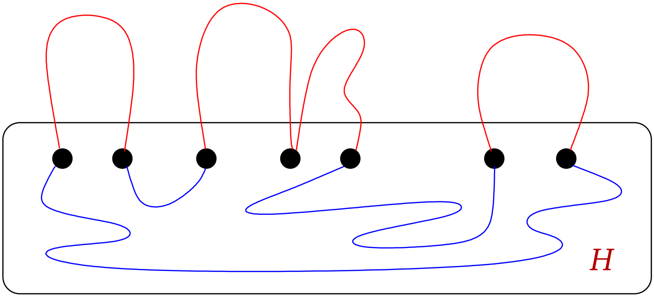

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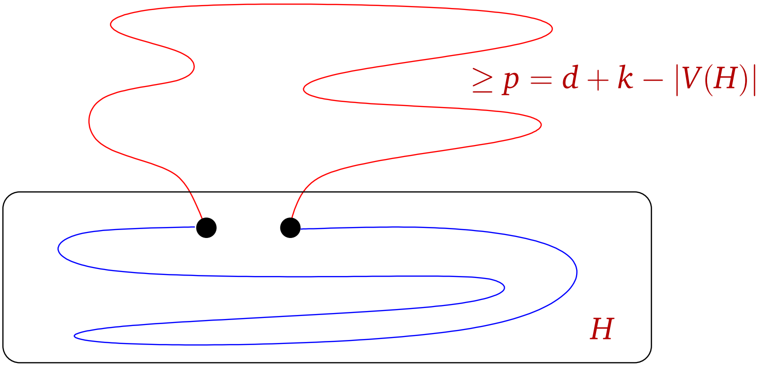

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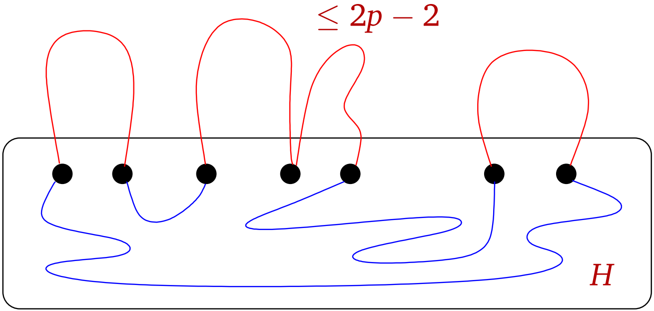

Tightness of the bound

Proposition

For any $\varepsilon > 0$*, it is* NP*-complete to decide whether a connected graph* G *contains a path with at least* $(1+\varepsilon)\mathrm{dg}(G)$ *vertices and it is* NP*-complete to decide whether a* 2*-connected graph* G *contains a cycle with at least* $(1+\varepsilon)\mathrm{dg}(G)$ *vertices.*

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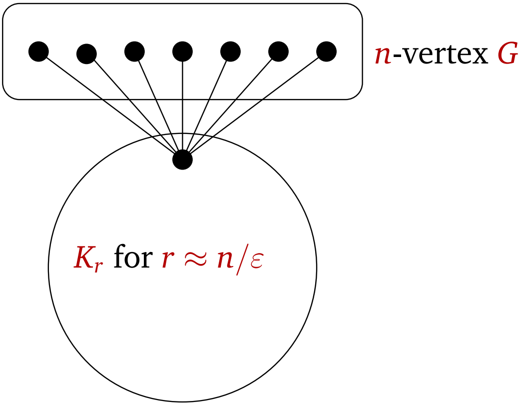

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- Longest Path Above Degeneracy is FPT for connected graphs when parameterized by k.
- Longest Cycle Above Degeneracy is FPT for 2-connected graphs when parameterized by k.
- For any $\varepsilon > 0$, it is NP-complete to decide whether a connected graph G contains a path with at least $(1+\varepsilon)\mathrm{dg}(G)$ vertices and it is NP-complete to decide whether a 2-connected graph G contains a cycle with at least $(1+\varepsilon)\mathrm{dg}(G)$ vertices.

Open problems

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Task: Decide whether G contains a path (cycle) with at least $ad(G) + k$ vertices.

Open problems

For a graph G, the average degree of G is

$$ad(G) = \sum_{v \in V(G)} d(v)/|V(G)| = 2|E(G)|/|V(G)|.$$

Observation: A graph G has a path with $ad(G) + 1$ vertices, and a 2-connected graph G contains a cycle with at least $ad(G) + 1$ vertices.

Input: A graph G and a positive integer k.

Task: Decide whether G contains a path (cycle) with at least $ad(G) + k$ vertices.

Is the problem FPT when parameterized by k for connected (2-connected) graphs?

Open problems

Exact Detour (for directed or undirected graphs)

→ randomized time 2.746^k

→ deterministic time 6.745^k

Detour (for undirected graphs)

→ deterministic time c^k

Open: Detour in directed graphs

Thank You!