Approximation Schemes for Geometric Coverage Problems

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Joint work with Steven Chaplick, Alexander Ravsky, and Joachim Spoerhase

Maximum Coverage

Given a ground set U, a set family $\mathcal{F} \subseteq 2^U$, and a positive integer k, find a family $\mathcal{S} \subseteq F$ of k sets maximizing the number $|\bigcup \mathcal{S}|$ of covered elements.

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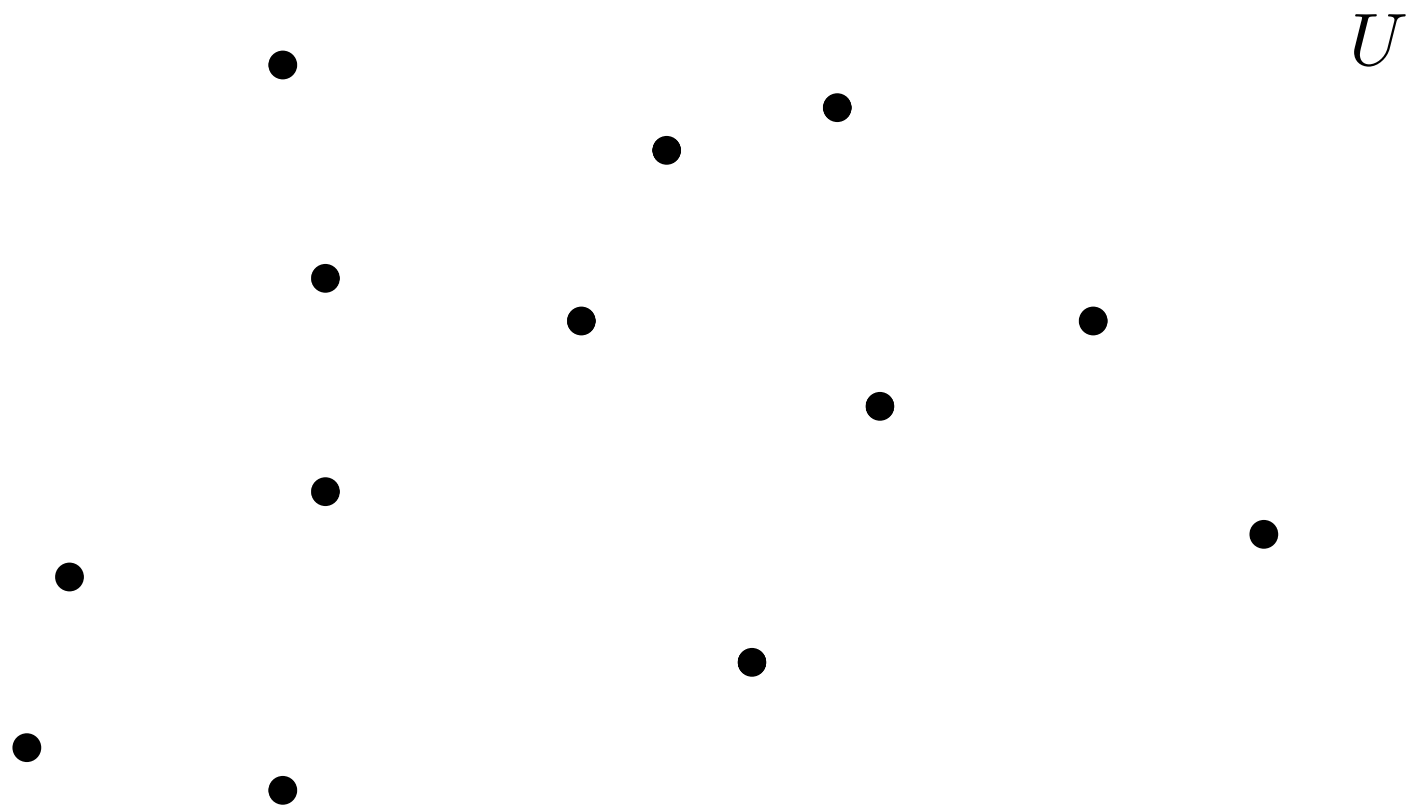

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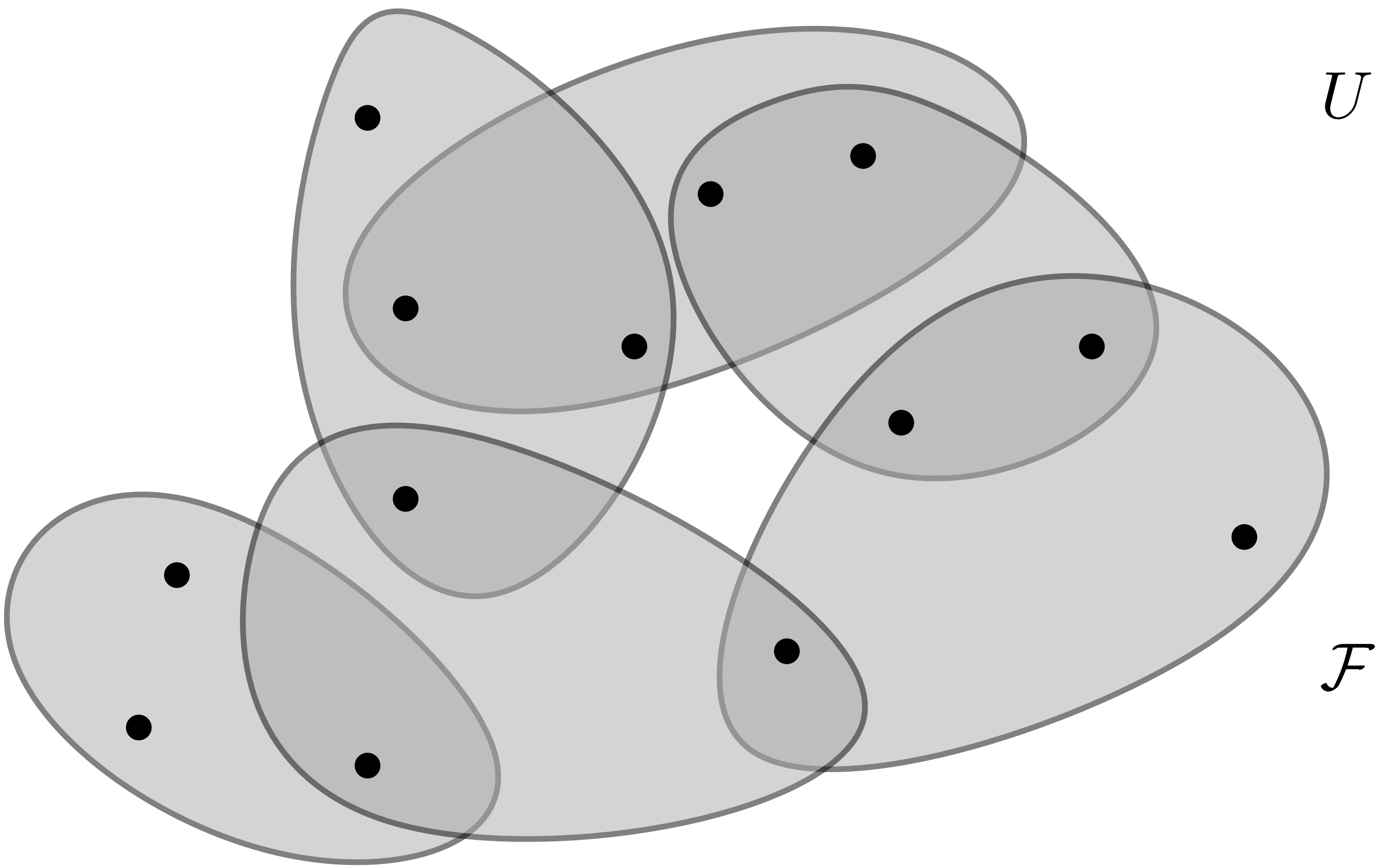

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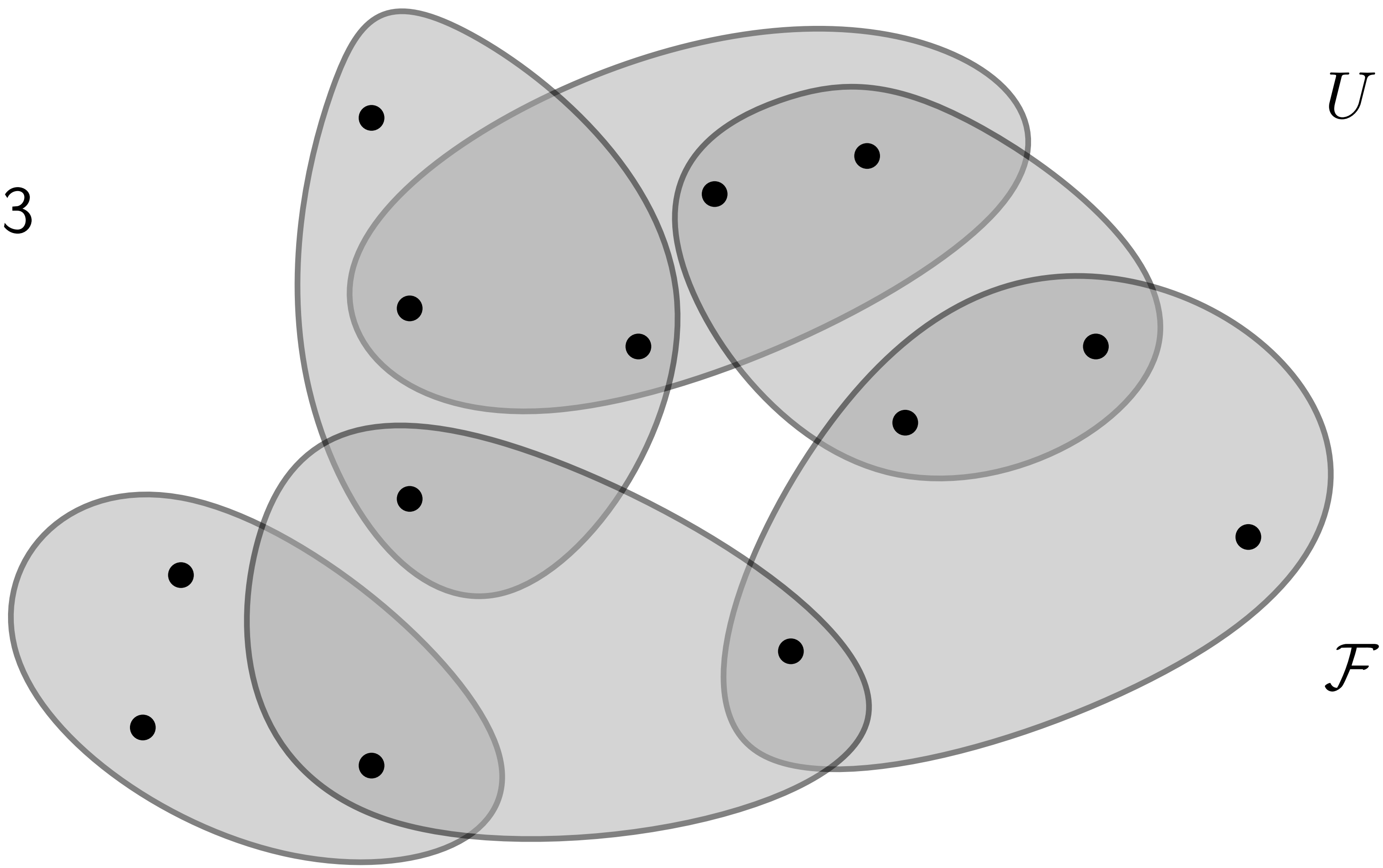

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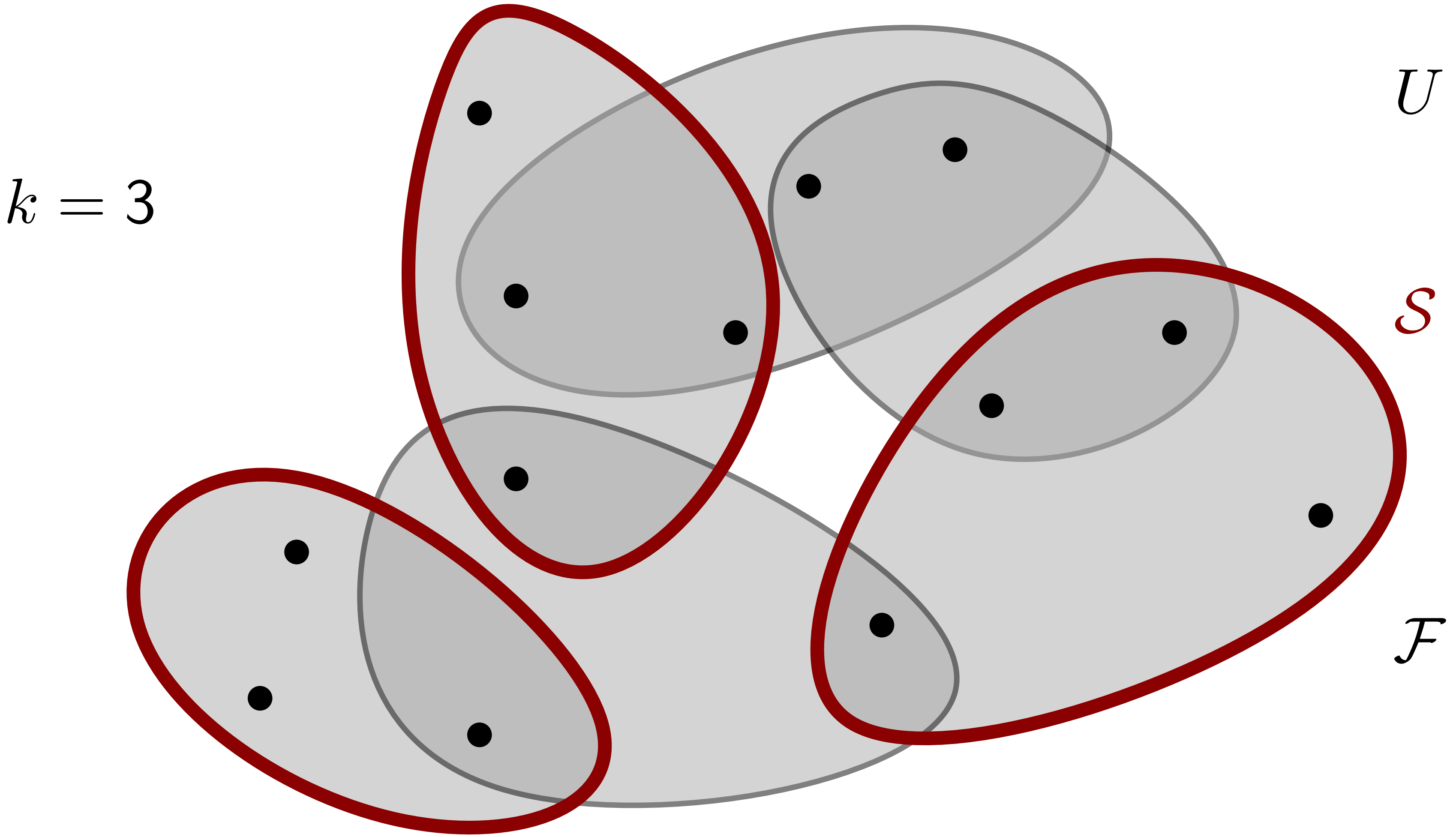

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Think of k-document search in information retrieval!

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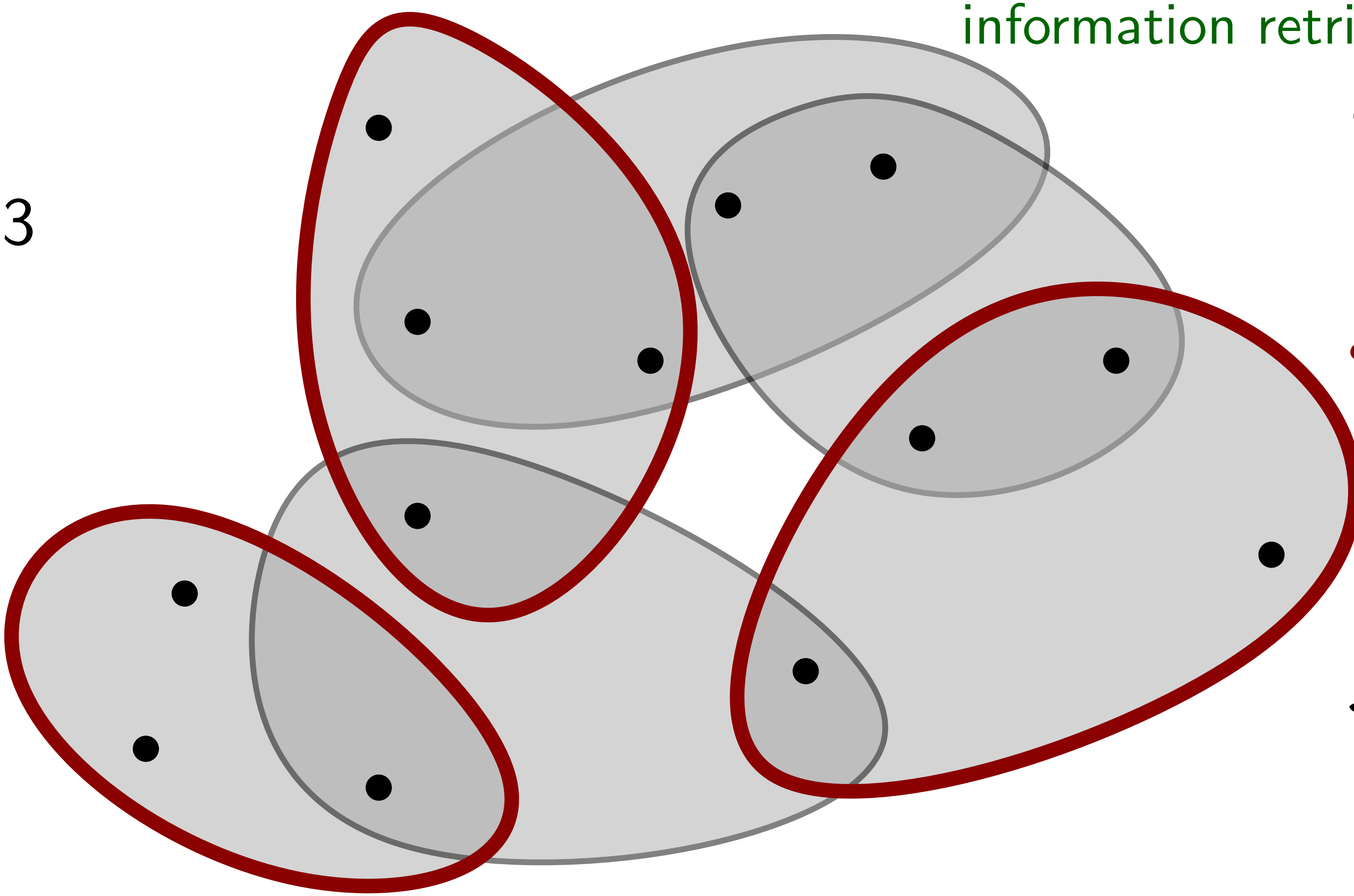

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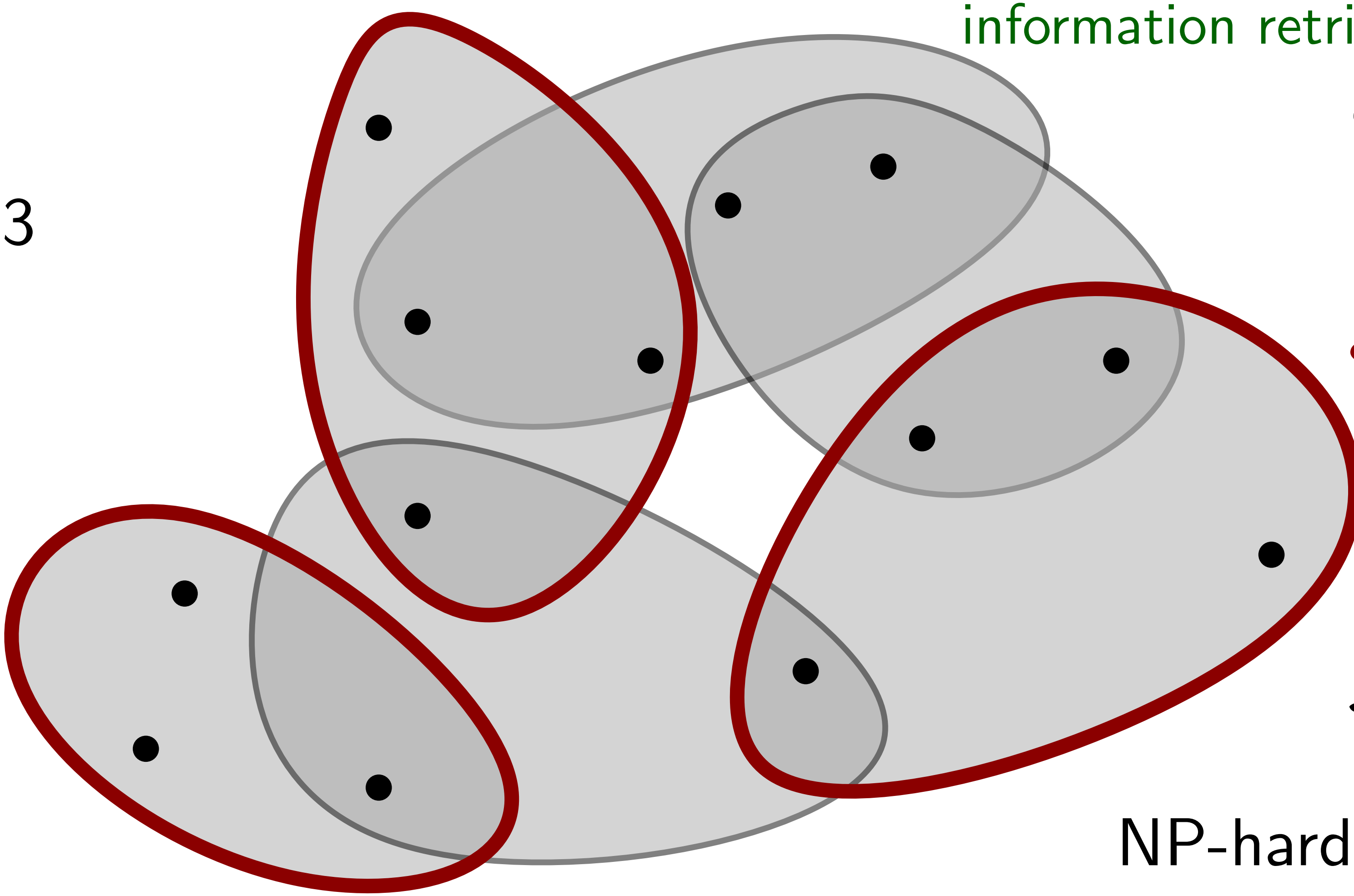

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NP-hard

Prior Work and Open Question

General Coverage

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- greedy gives a $(1 - 1/e)$-approximation
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Question: In which of the geometric cases that are not known to be APX-hard (e.g. halfspaces in $\mathbb{R}^3$, pseudodisks in $\mathbb{R}^2$, ...) can we obtain a (true) PTAS?

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Geometric Set Cover

- many techniques and a large body of literature: ϵ-nets, quasi-uniform sampling and many more ...
- local search gives a PTAS for a multitude of problems: halfspaces in $\mathbb{R}^3$, pseudodisks in $\mathbb{R}^2$, terrain guarding,... [Mustafa & Ray 2009]

Geometric Set Cover and Local Search

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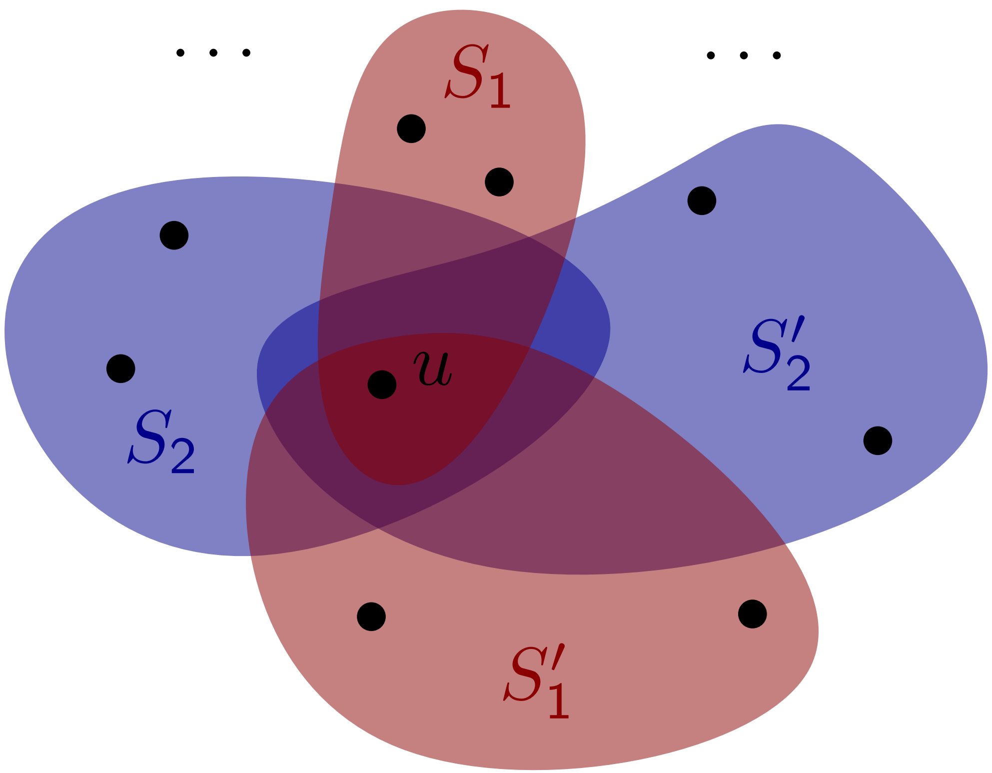
planar exchange graph → Set Cover / **Max Coverage?** → PTAS

Exchange Graph

[Mustafa & Ray 2009]

Let $\mathcal{F}_1, \mathcal{F}_2$ be feasible solutions. A graph with node set $\mathcal{F}_1, \mathcal{F}_2$ has the **exchange property** if for every $u \in U$ covered by both solutions there exist an edge (S_1, S_2) with $u \in S_1 \cap S_2$ and $S_i \in \mathcal{F}_i$



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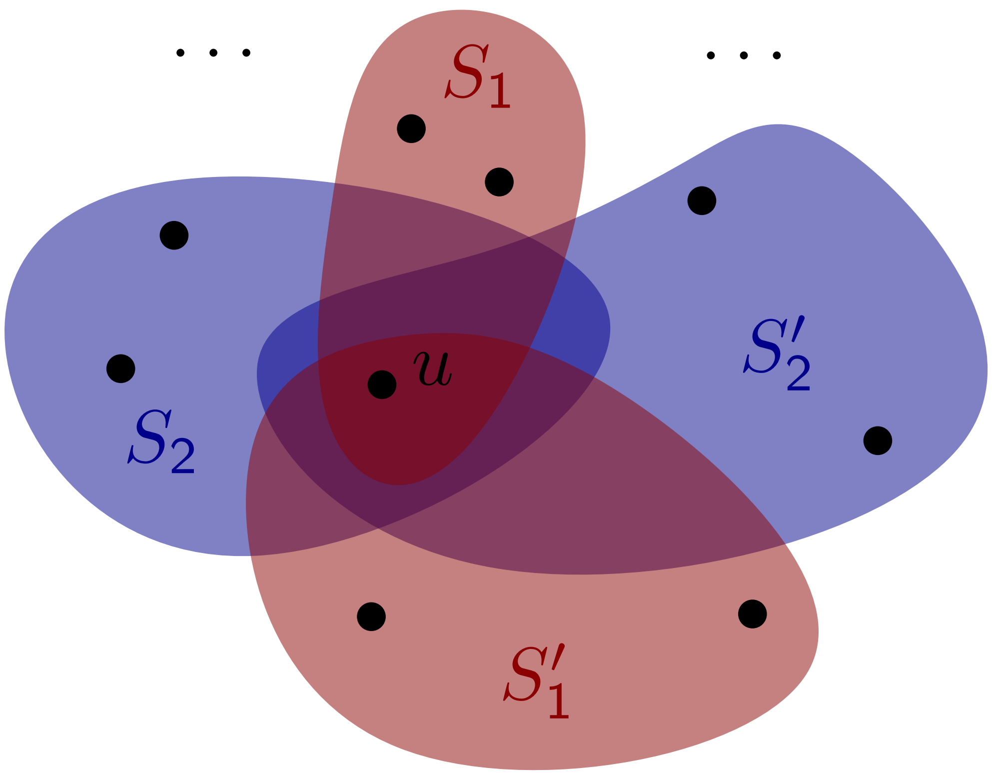

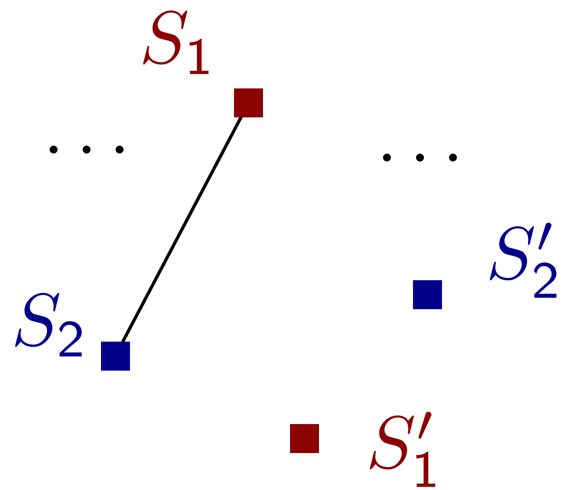

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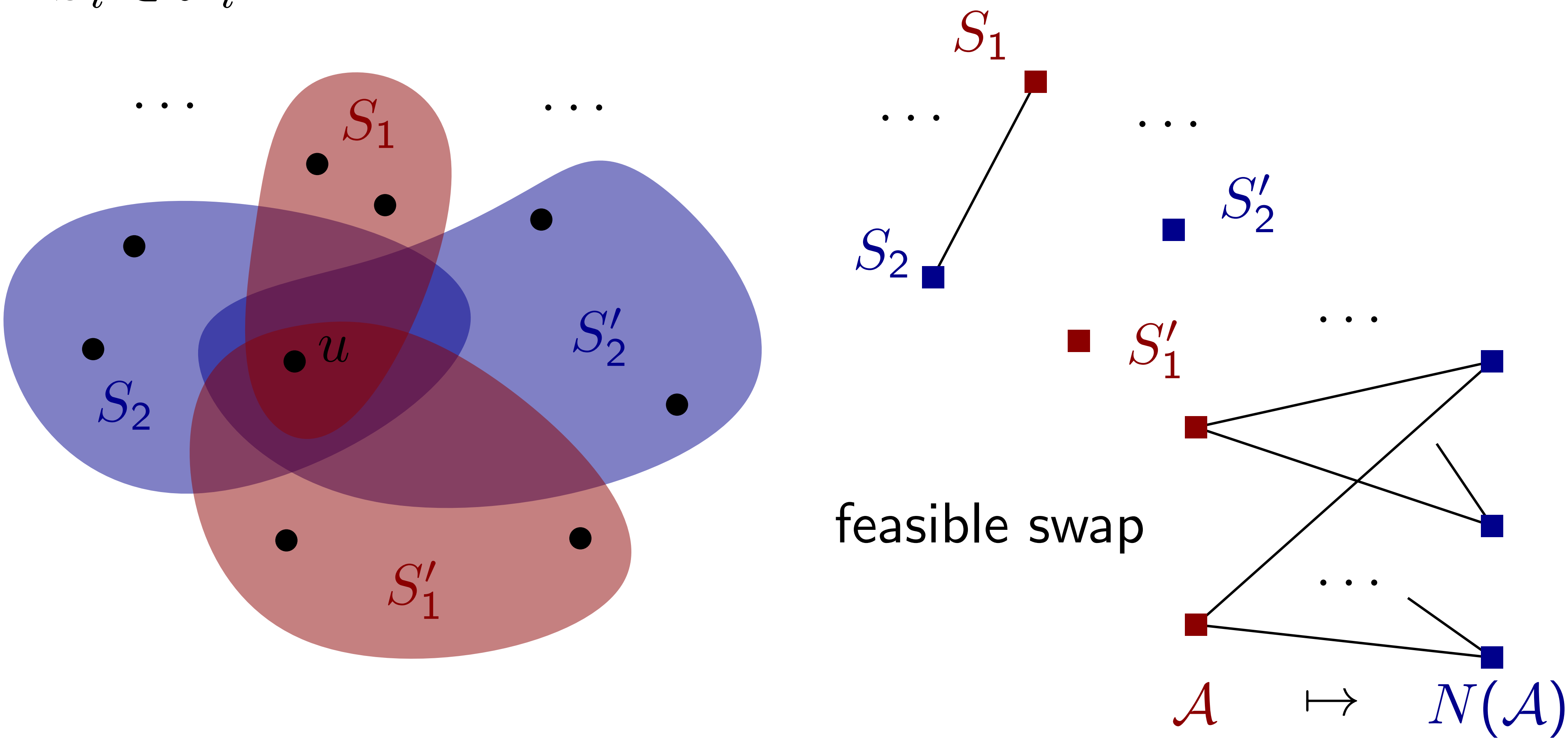

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Theorem (Frederickson 1987):

For every $t > 0$, planar G, $V(G)$ partitions as $(V_1, \ldots, V_\ell, X)$, $\ell = \Theta(n/t)$ such that

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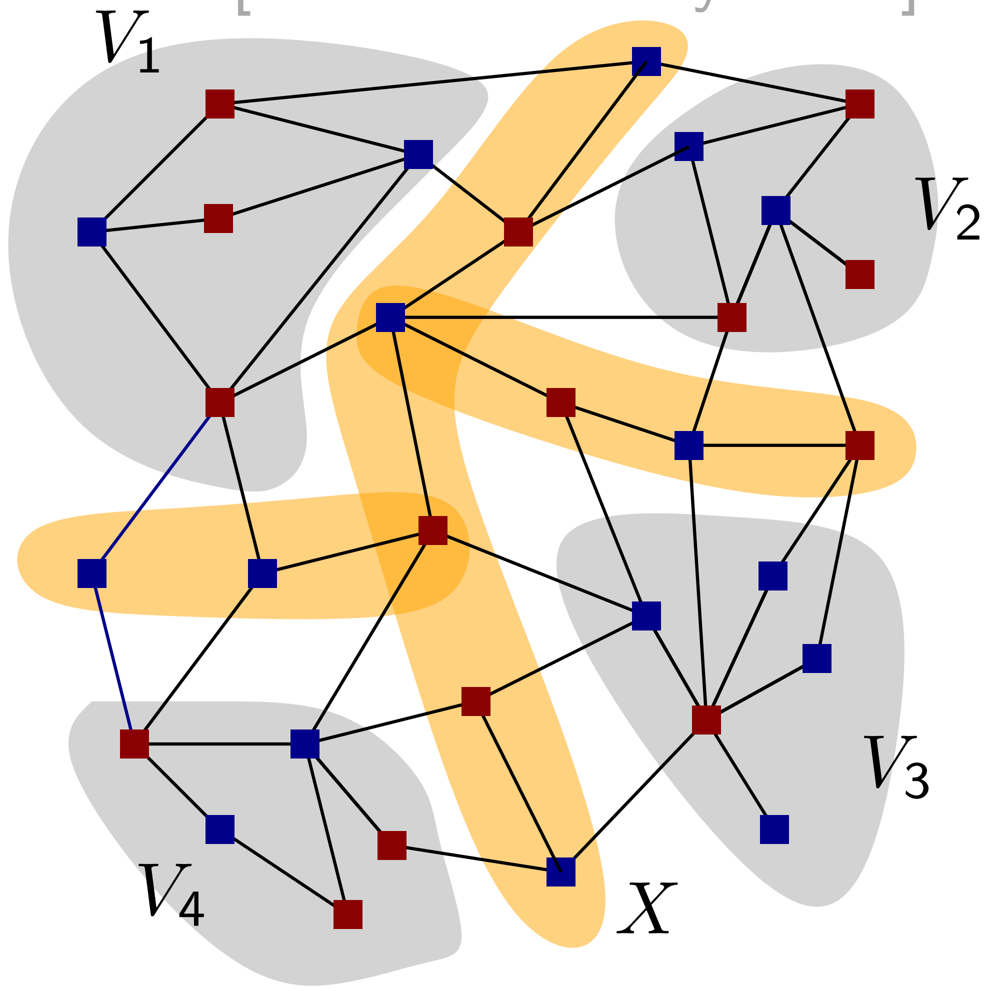

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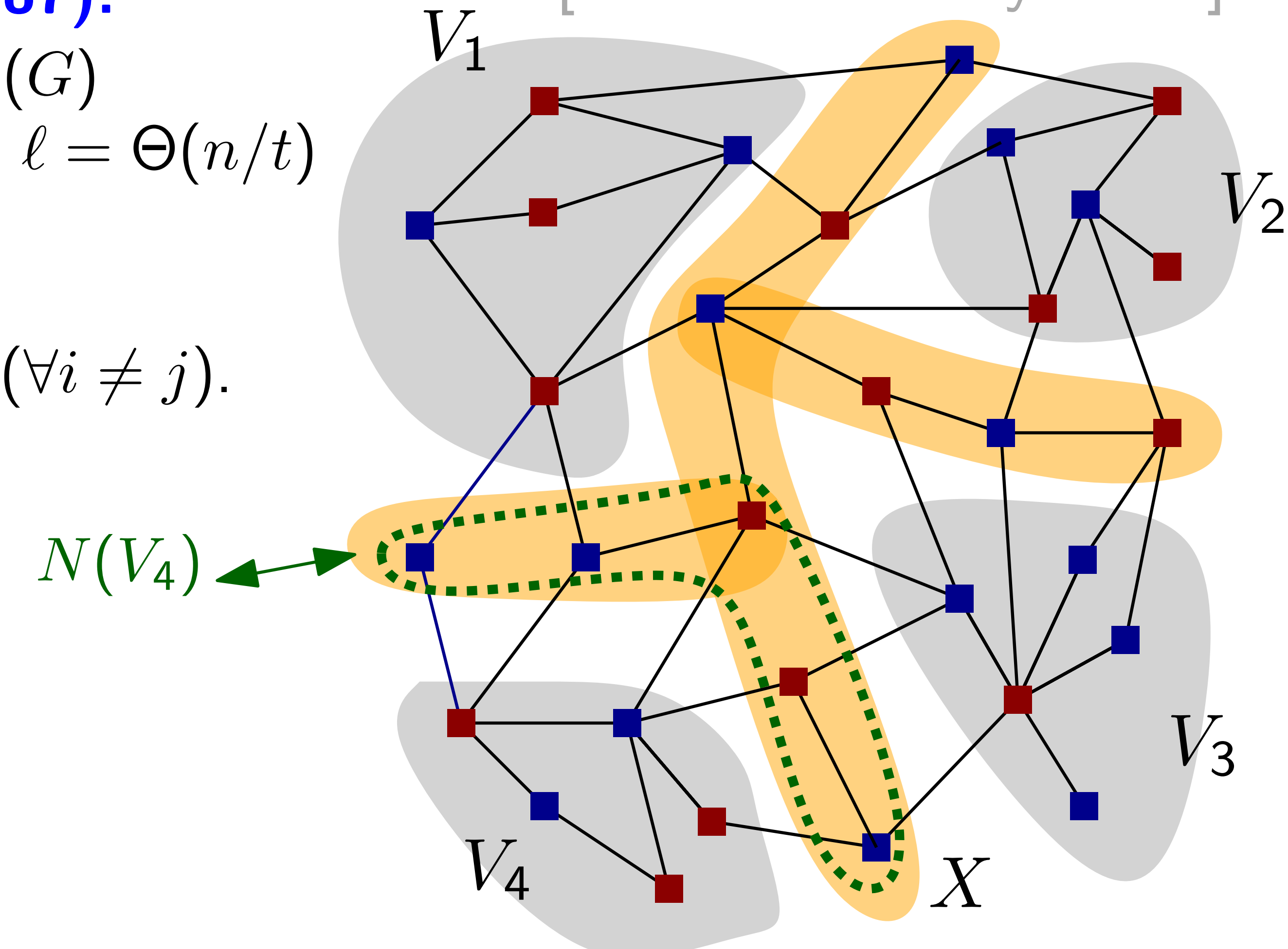

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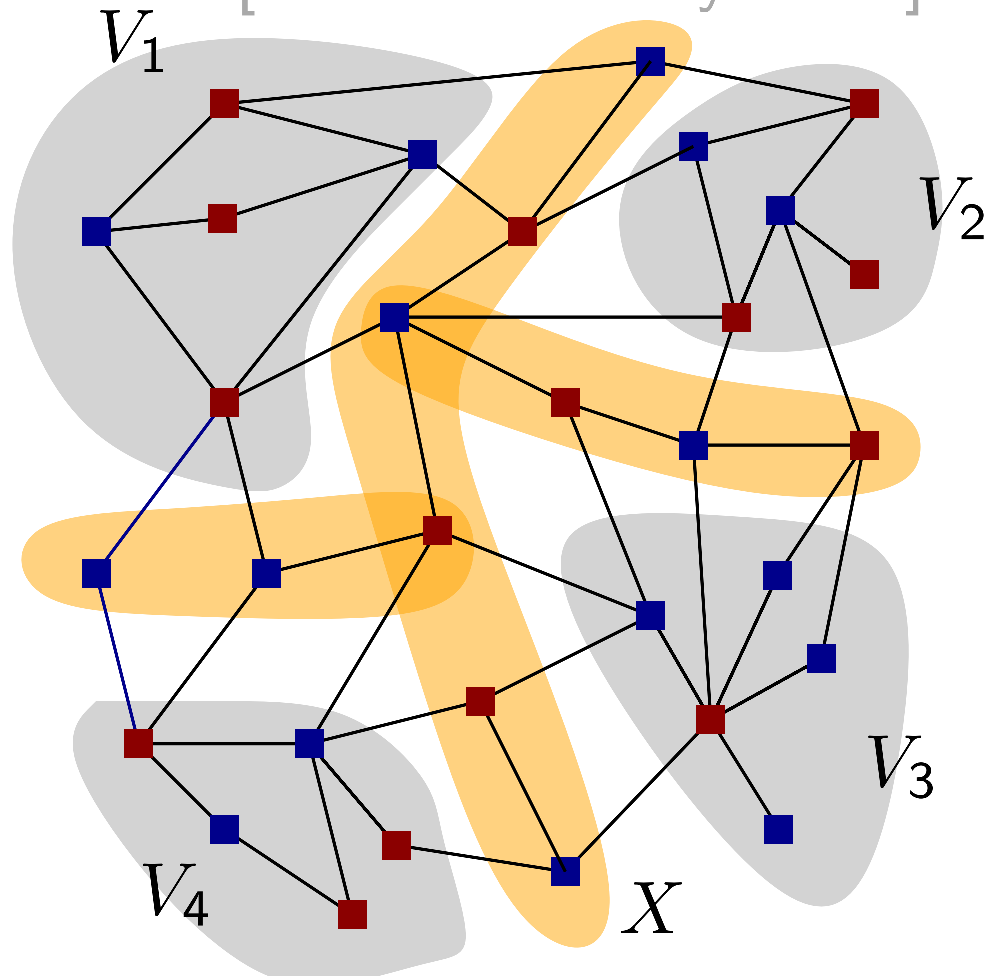

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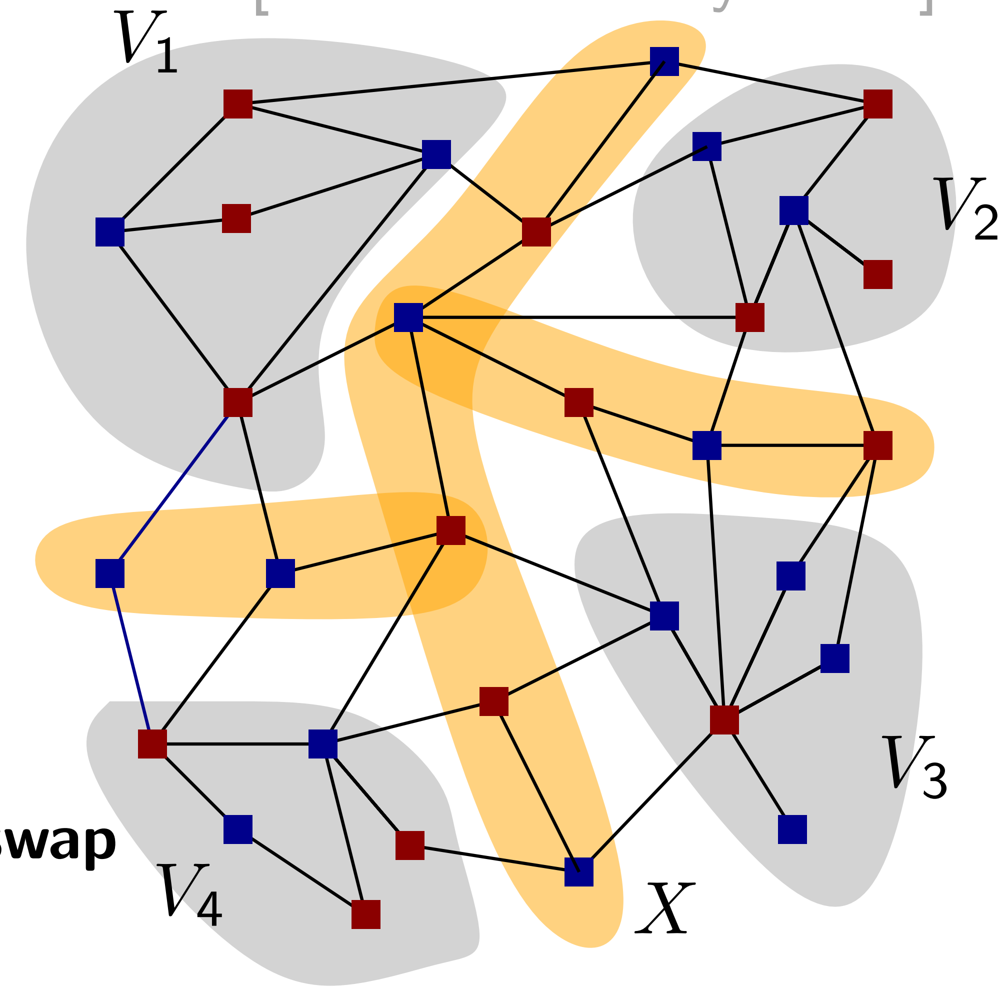

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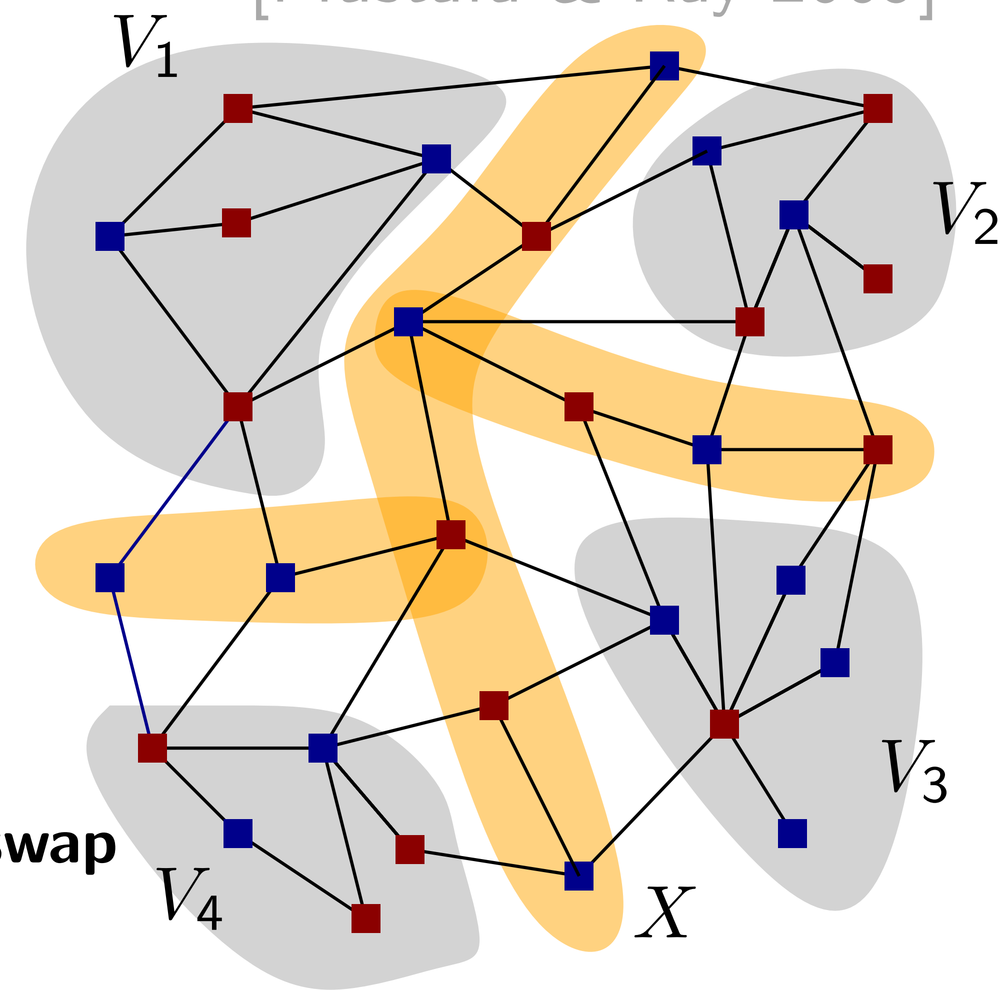

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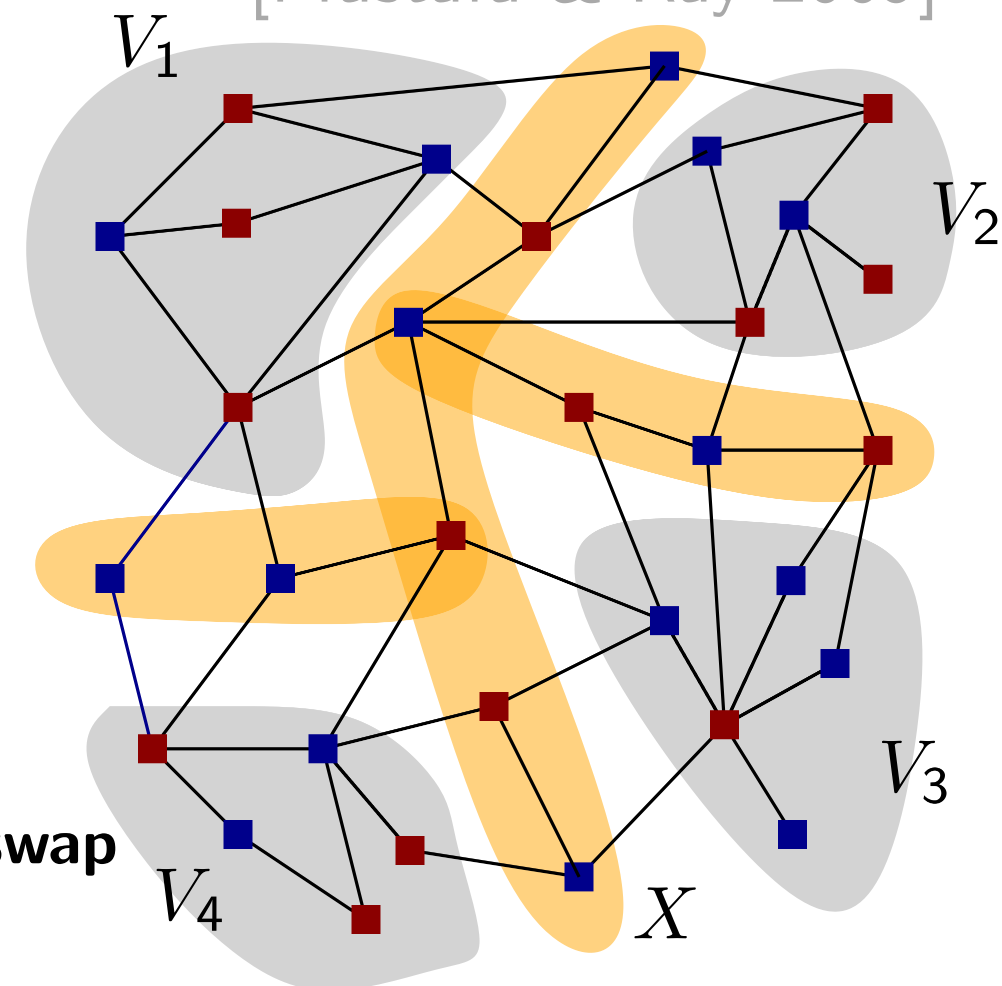

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[this work]

local search: swaps **do not change cardinality** of the solution but **improve number of covered elements**

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2. color-imbalanced subdivisions conflict with **hard cardinality constraint**

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Corollary: Max coverage admits a PTAS for

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- hitting pseudodisks (r-admissable regions) in $\mathbb{R}^2$ by points
- guarding 1.5D terrains
- maximum k-dominating set for intersection graphs of homethetic copies of convex objects (such as arbitrary squares, translated and scaled copies of convex objects)
- maximum k-dominating set on non-trivial minor-closed graph classes
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confirms conjecture by Badanidiyuru, Kleinberg, Lee 2012

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Theorem: For every $t > 0$, planar two-colored G with $V(G) = R \cup B$ and $|B| = |R|$, $V(G)$ partitions as $(V_1, \ldots, V_\ell, X)$, $\ell = \Theta(n/t)$ such that

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3. Use **submodularity** to get a **perfectly balanced** and (still) **profitable** swap

Obtaining Approximate Color-Balance

[this work]

Step 1.1: **Uniform** subdivision

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Obtaining Approximate Color-Balance

[this work]

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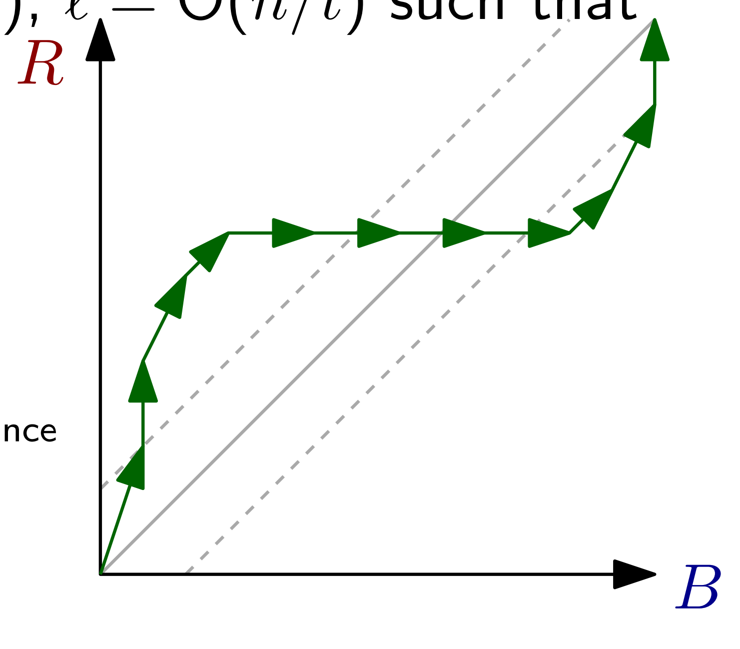

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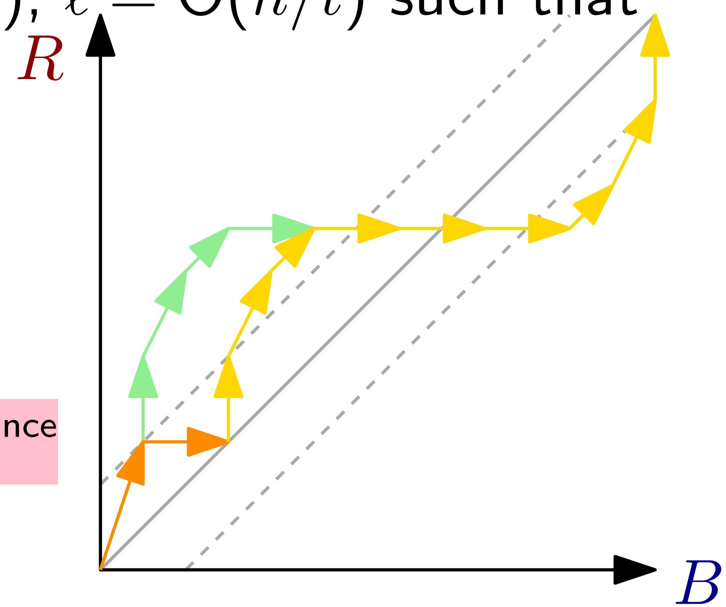

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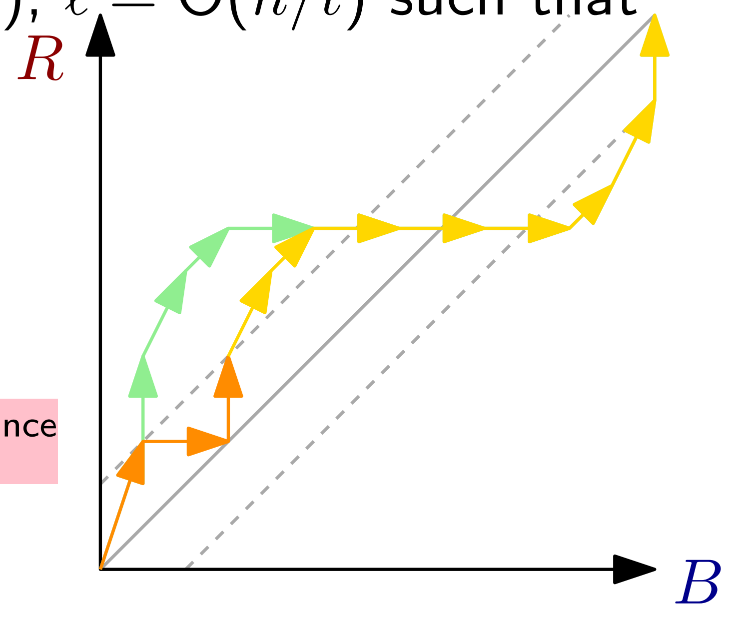

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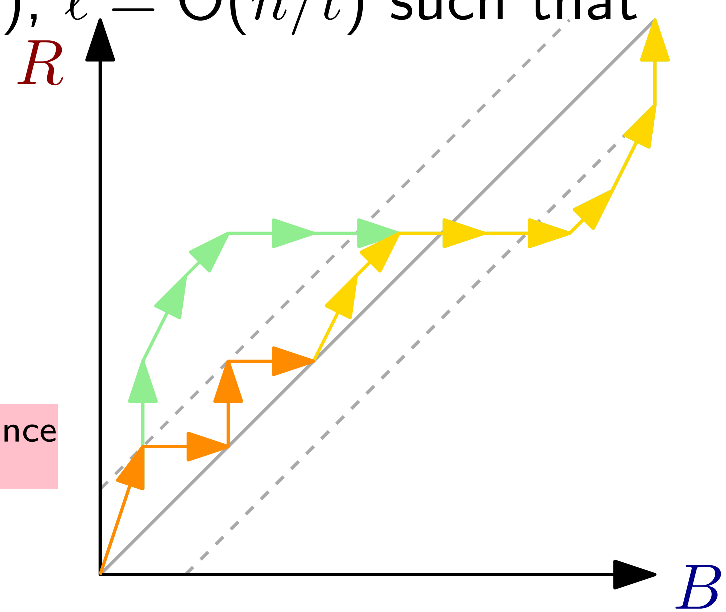

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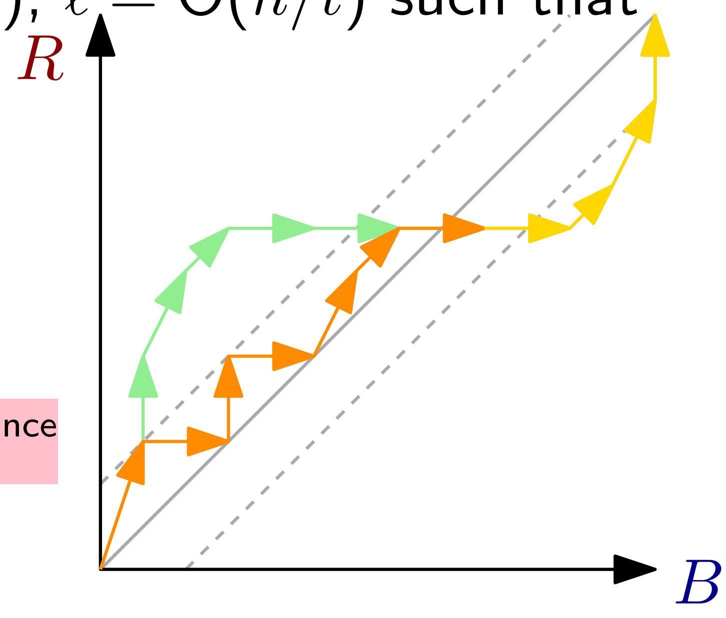

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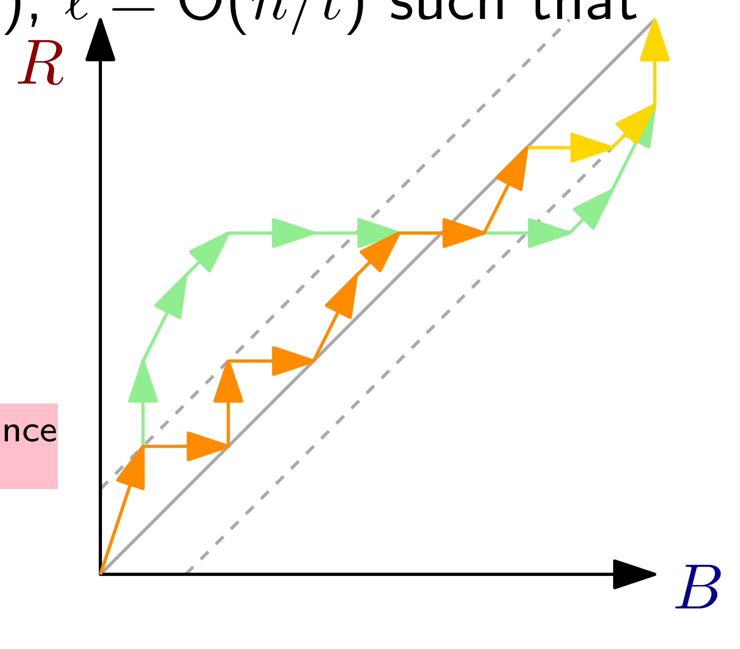

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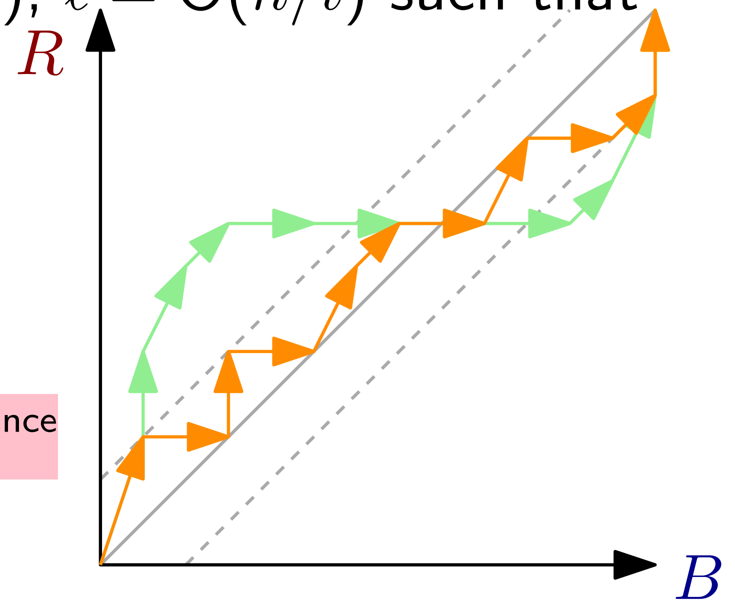

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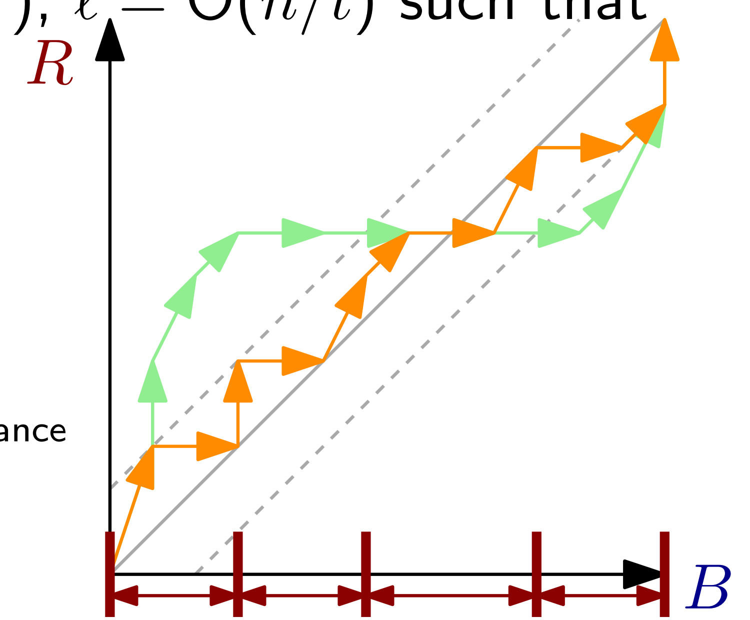

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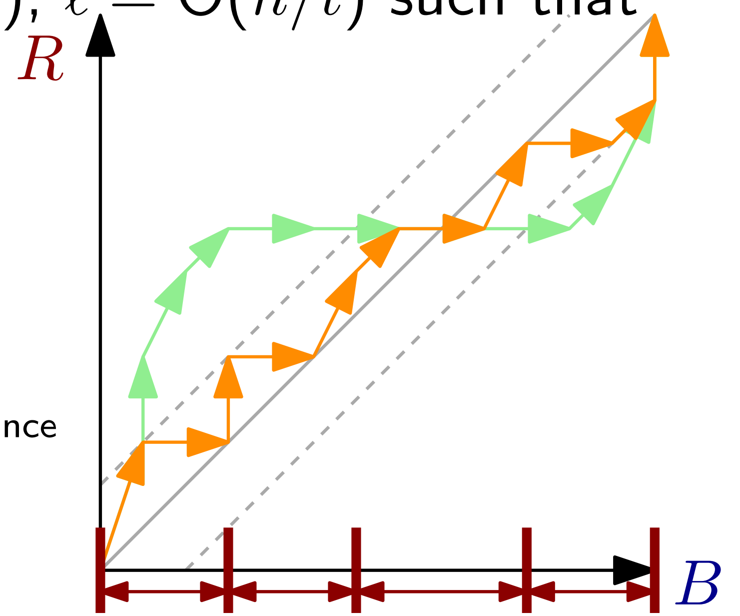

From Nearly Balanced to Balanced Swaps

[this work]

Step 2: Obtain a **significantly profitable, almost balanced** swap

$$\min_i \frac{|L_i|}{|W_i|} \leq \frac{\sum_i |L_i|}{\sum_i |W_i|} \leq \frac{\mathsf{ALG} - |Z|}{\mathsf{OPT} - |Z|} \leq \frac{\mathsf{ALG}}{\mathsf{OPT}}$$

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elements **lost** by swap out

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1. swap out $V_i \cap \mathcal{S}$
2. swap in $(V_i \cup N(V_i)) \cap \mathcal{O}$

elements **won** by swap in

elements staying covered during swaps

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$\rightsquigarrow$ ex. i with $|L_i| < (1 - c/\sqrt{b})|W_i|$

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S_1 maximizes $|S \setminus Z_i|$

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$\leadsto$ **balanced and profitable swap**

Overview and Future Work

Theorem: There is a PTAS for any class of max coverage problems that admits planar (f-separable) exchange graphs.

Corollary: Max coverage admits a PTAS for

confirms conjecture by Badanidiyuru, Kleinberg, Lee 2012

- covering points with halfspaces in $\mathbb{R}^3$
- covering points with pseudodisks in $\mathbb{R}^2$
- hitting pseudodisks (r-admissable regions) in $\mathbb{R}^2$ by points
- guarding 1.5D terrains
- maximum k-dominating set for intersection graphs of homethetic copies of convex objects (such as arbitrary squares, translated and scaled copies of convex objects)
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- improve running time
- improved ratios for APX-hard cases?
- other applications (with hard cardinality constraint)?

Thank you!