The Maximum Distance-d Independent Set Problem on Unit Disk Graphs

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Organization of Talk

Preliminaries

1. Unit Disk Graph
2. Independent Set
3. Distance-d Independent Set

Unit Disk Graph (UDG)

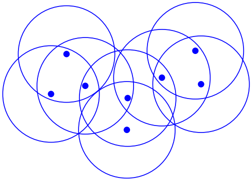


Figure: A collection of unit disks

Unit Disk Graph (UDG)

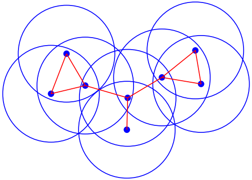


Figure: The corresponding unit disk graph

Unit Disk Graph (UDG)

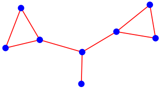


Figure: The final graph without disks

Independent set

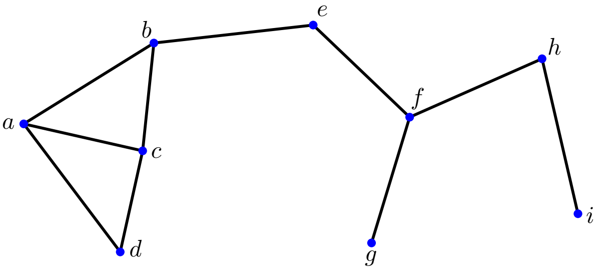


Figure: General graph

Independent set

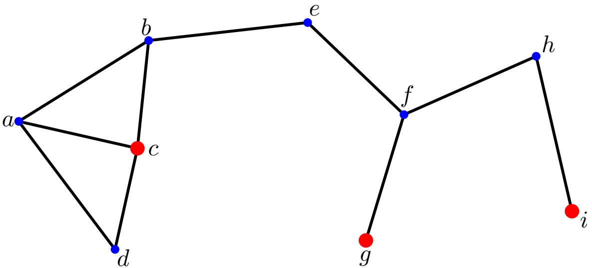


Figure: Example of independent set in general graph

Independent set

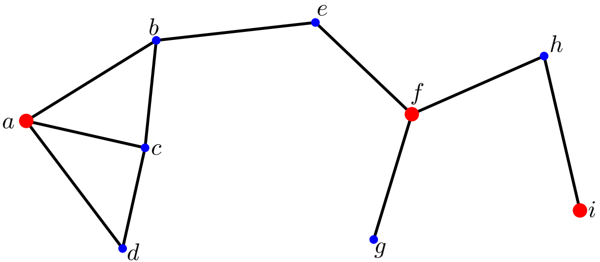


Figure: Example of a different independent set

Maximum independent set

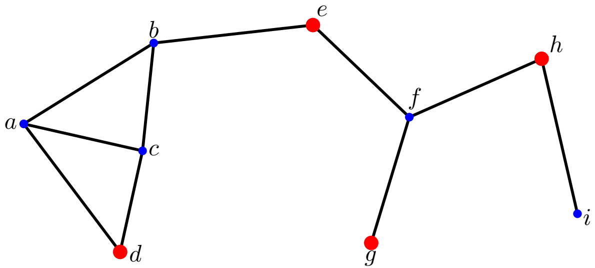


Figure: Example of a maximum independent set

Maximum independent set

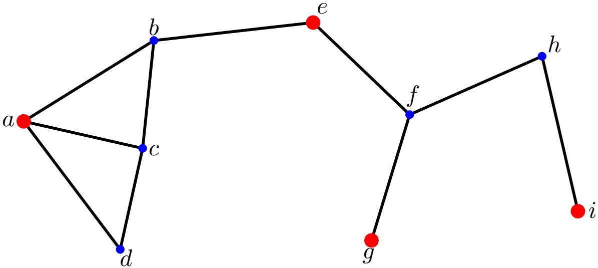


Figure: Example of a different maximum independent set

Distance-d Independent Set

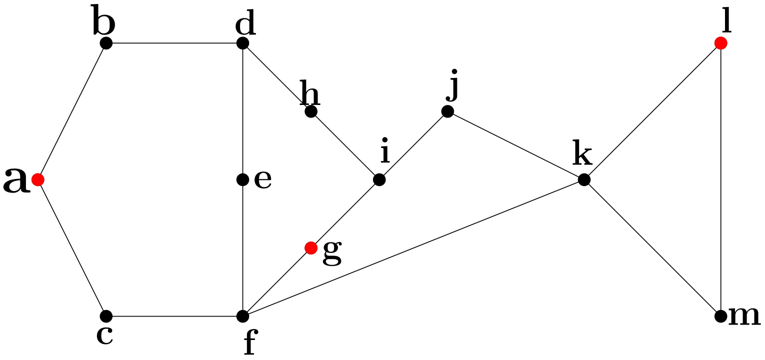


Figure: Distance-d independent set for $d = 3$

Distance-d Independent Set

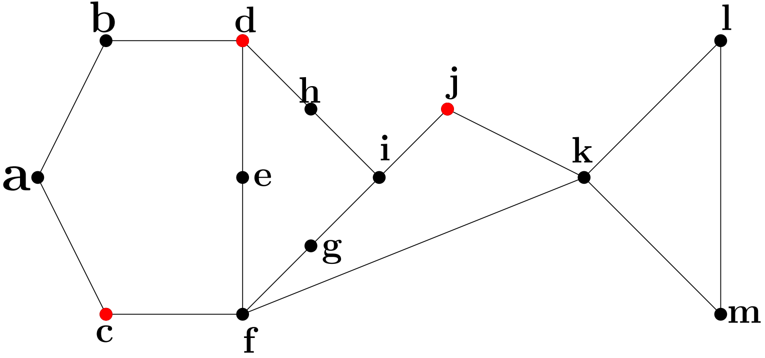


Figure: Distance-d independent set for $d = 3$

Maximum Distance-d Independent Set

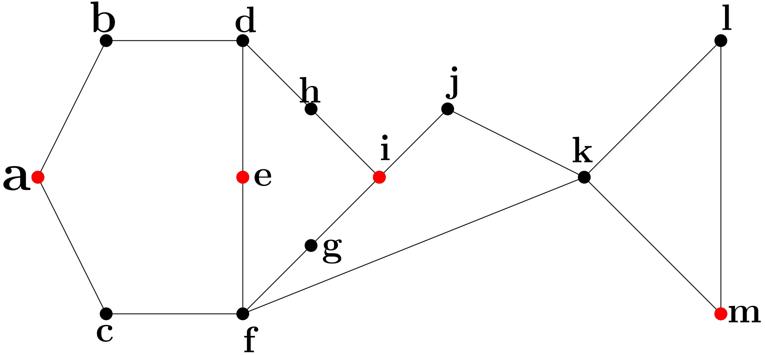


Figure: Maximum distance-d independent set for $d = 3$

Maximum Distance-d Independent Set

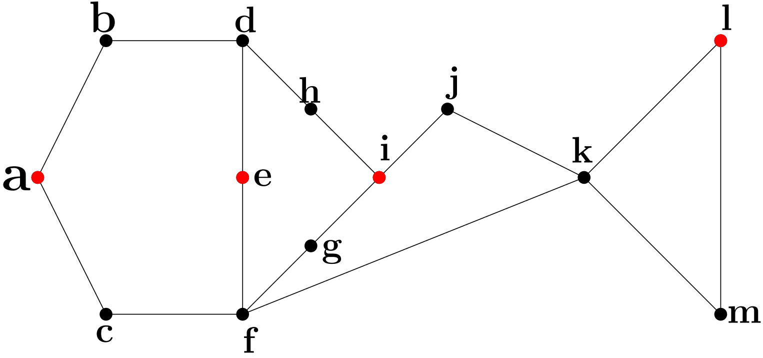


Figure: Maximum distance-d independent set for $d = 3$

The GMDdIS Problem on Unit Disk Graphs

Given: A unit disk graph $G = (V, E)$ corresponding to a point set $P = \{p_1, p_2, \ldots, p_n\}$ in the plane.
Objective: A maximum cardinality subset $I \subseteq V$, such that for every pair of vertices $p_i, p_j \in I$, the length (number of edges) of the shortest path between p_i and p_j is at least d.

Related Work

Independent set

1. NP-hard, [Clark et al., 1990].
2. 5-factor and 3-factor approximation algorithms, [Marathe et al., 1995].
3. PTAS in time $n^{O(k^2)}$, [Hunt et al., 1998].
4. First PTAS for MWIS in $n^{O(k^2)}$ time, [Erlebach et al., 2005].
5. A 2-f.a.a and a PTAS in $O(n^3)$ and $n^{O(k \log k)}$ time resp., [Das et al., 2015].
6. A 2-f.a.a and a PTAS in $O(n^2 \log n)$ and $n^{O(k)}$ time resp., [Jallu and Das, 2016].

The GMDdIS Problem on Unit Disk Graphs

1. we study the MDdIS problem on unit disk graphs and we call it *the geometric maximum distance-d independent set* (GMDdIS) problem.
2. We show that the decision version of the GMDdIS problem (for $d \geq 3$) is NP-complete on unit disk graphs.
3. We propose a 4-factor approximation algorithm for the problem.
4. We also proposed a PTAS for this problem.

Our work

NP-complete

1. GEOMETRIC DISTANCE-d INDEPENDENT SET (GDdIS) PROBLEM

   Instance : An unweighted unit disk graph $G = (V, E)$ defined on a point set P and a positive integer $k \leq |V|$.

   Question : Does there exist a distance-d independent set of size at least k in G?

2. DISTANCE-d INDEPENDENT SET ON PLANAR BIPARTITE GRAPHS[Eto et al. (2014)]

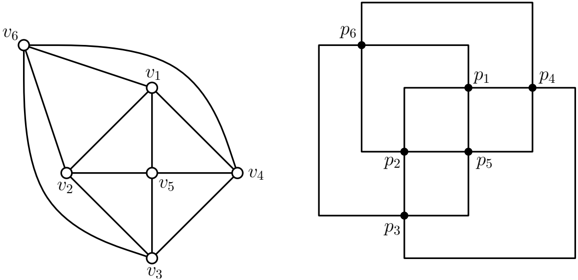
   Instance : An unweighted planar bipartite graph $G = (V, E)$ with girth at least d and maximum vertex degree 3, and a positive integer $k \leq |V|$.

   Question : Does there exist a distance-d independent set of size at least k in G?

Planar embedding of planar graphs

Key lemma [Valiant, 1981]

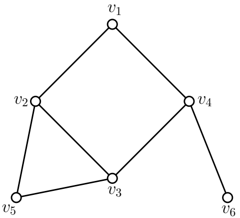
Planar graph $G = (V, E)$ with maximum degree 4 can be embedded in the plane using $O(|V|^2)$ area in such a way that its vertices are at integer co-ordinates and its edges are drawn so that they are made up of line segments of the form $x = i$ or $y = j$, for integers i and j.

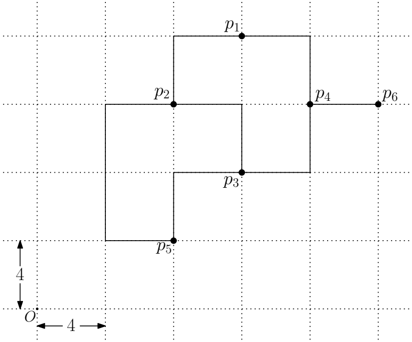


Observation

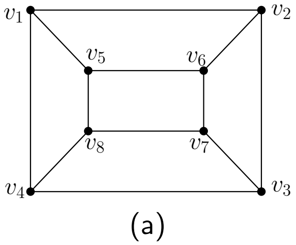
Corollary

Let $G = (V, E)$ be a planar graph with maximum degree 3. G can be embedded in the plane such that its vertices are at $(4i, 4j)$ and its edges are drawn as a sequences of consecutive line segments drawn on the lines $x = 4i$ or $y = 4j$, for some integers i and j.





NP-hardness proof for DdIS problem



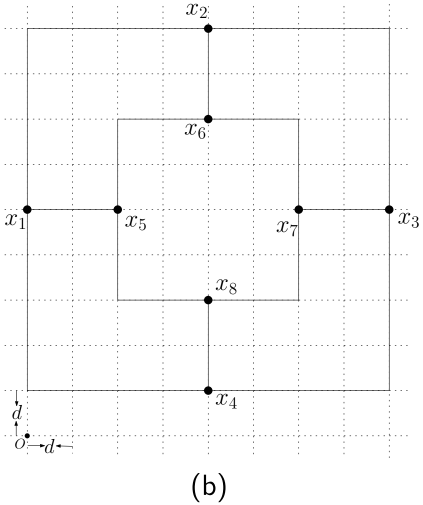
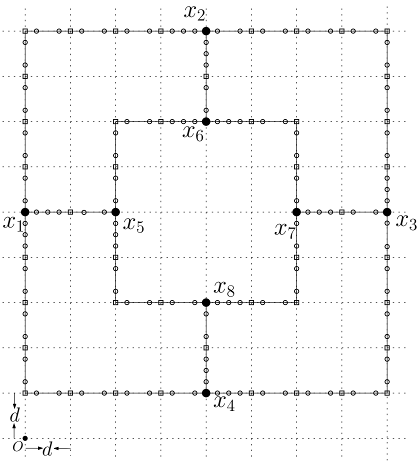
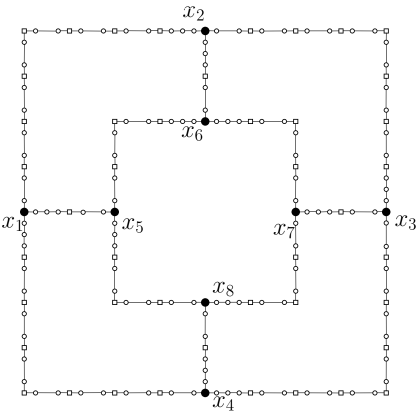


Figure: (a) A planar bipartite graph G of maximum degree 3, (b) its embedding G' on a grid of cell size 3×3.

NP-hardness proof for DdIS problem



(c)



(d)

Figure: (c) adding of extra points to G', (d) the obtained UDG G''.

NP-hardness proof for DdIS problem

Claim

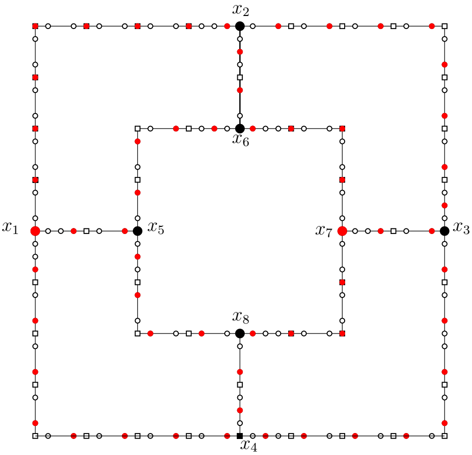
G has a DdIS of cardinality at least k if and only if G'' has a DdIS of cardinality at least $k + \ell$.

NP-hardness proof for DdIS problem

Lemma

Any DdIS of G'' contains at most ℓ points from segments.

NP-hardness proof for DdIS problem



The obtained UDG G''.

Approximation algorithm for DdIS problem

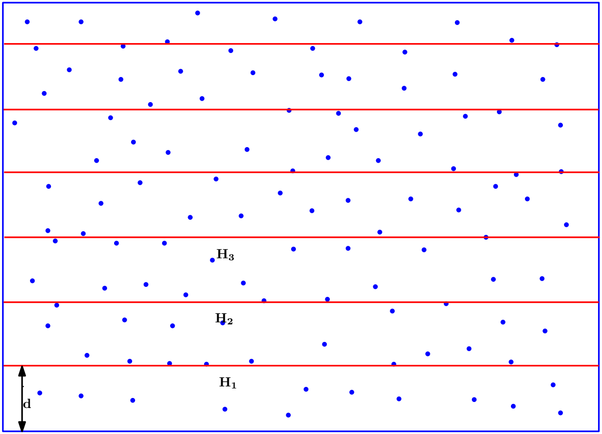


Figure: The region divided with horigental strips of width d

Approximation algorithm for DdIS problem

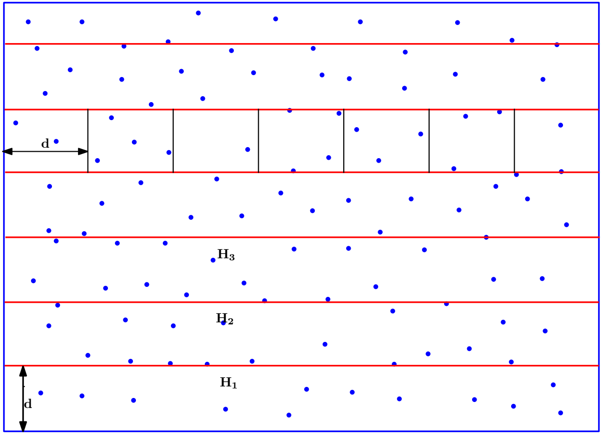


Figure: Each strip divided with vertical strips of width d

Approximation algorithm for DdIS problem

Lemma

If OPT is an optimum solution for the given GMDdIS problem, then $\max(|S_{even}|, |S_{odd}|) \geq \frac{1}{4}|OPT|$.

Computing an optimum solution in a $d \times d$ square

- Let $\mathcal{Q} \subseteq P$ be the set of points inside a $d \times d$ square χ.
- G_χ be the UDG defined on $\mathcal{Q}$
- Let $C_1, C_2, \ldots, C_l$ be the connected components of G_χ.
- Without loss of generality we assume that any two components in G_χ are at least d distance apart[1] in G.

[1] if there are two components having distance less than d in G, then we can view them as a single component

Computing an optimum solution in a $d \times d$ square

Lemma

The worst case number of components in G_χ is $O(d^2)$.

Computing an optimum solution in a $d \times d$ square

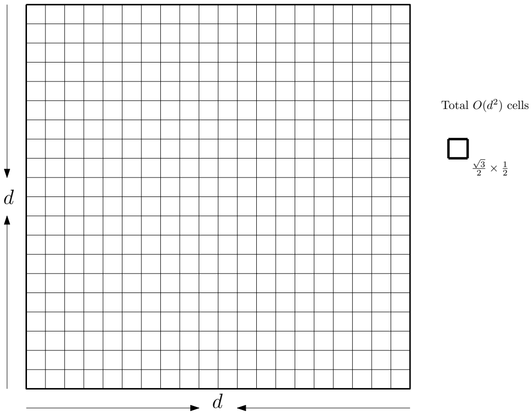


Figure: Maximum number of components in a $d \times d$ square

Computing an optimum solution in a $d \times d$ square

Lemma

Let C be any component of G_χ. The number of mutually distance-d independent points in C is bounded by $O(d)$.

Computing an optimum solution in a $d \times d$ square

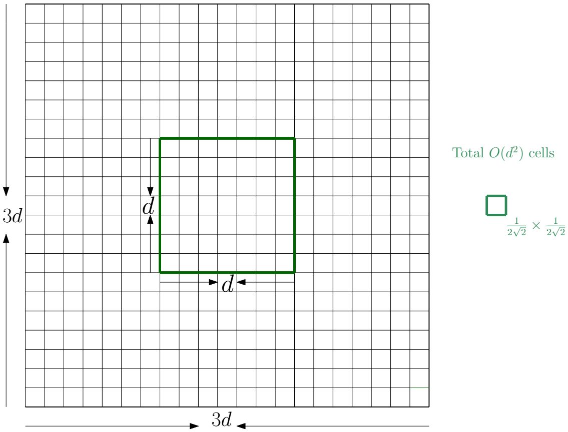


Figure: Maximum number of mutually distance-d independent set in a $d \times d$ square

Computing an optimum solution in a $d \times d$ square

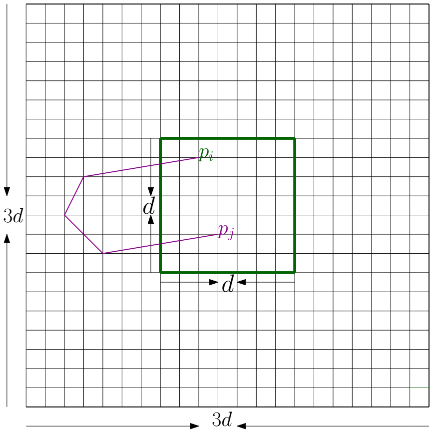


Figure: Maximum number of mutually distance-d independent set in a $d \times d$ square

Computing an optimum solution in a $d \times d$ square

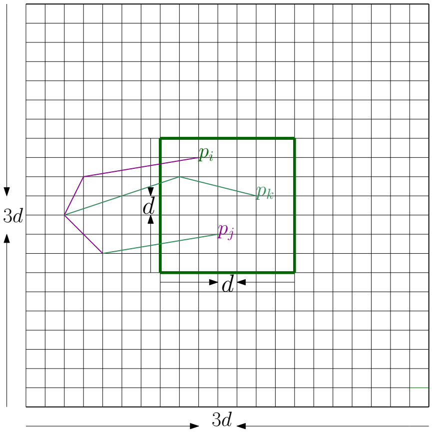


Figure: Maximum number of mutually distance-d independent set in a $d \times d$ square

Computing an optimum solution in a $d \times d$ square

Lemma

An optimal DdIS in χ can be computed in $d^2 n^{O(d)}$ time.

Computing an optimum solution in a $d \times d$ square

Theorem

Given a set P of n points in the plane, we can always compute a DdIS of size at least $\frac{1}{4}|OPT|$ in $d^2 n^{O(d)}$ time, where $|OPT|$ is the cardinality of a GMDdIS.

Approximation scheme for DdIS problem

We use two level shifting strategy

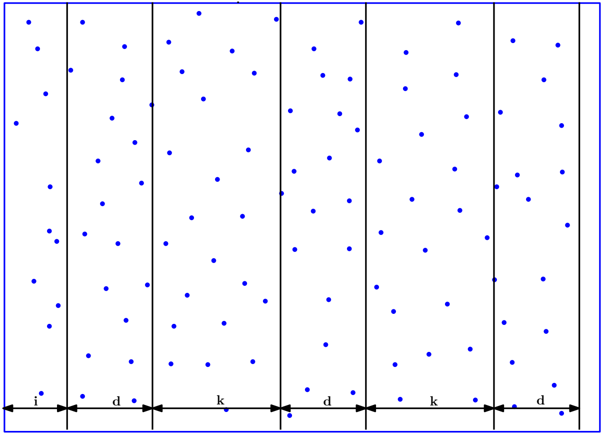


Figure: On i^{th} iteration first vertical strip is of width i, all the even strip is of width$d$ and all the odd strips is of width k, where $k >> d$.

Approximation scheme for DdIS problem

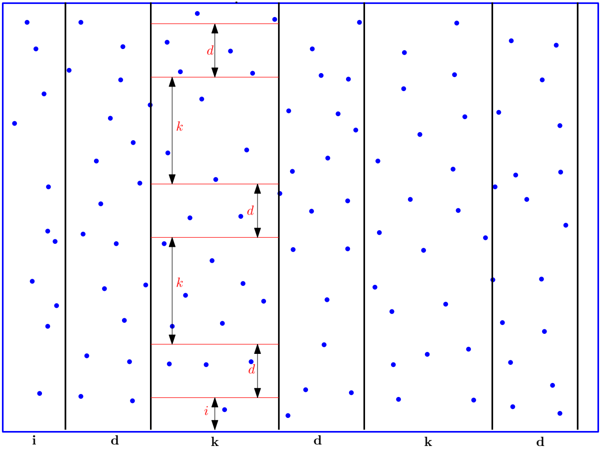


Figure: Each vertical odd strip use the same shifting strategy

Approximation scheme for DdIS problem

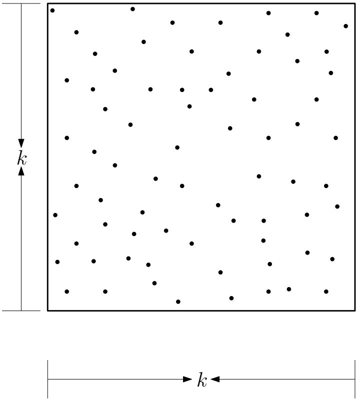


Figure: $\mathcal{Q} \subseteq P$ inside a square χ of size $k \times k$.

Approximation scheme for DdIS problem

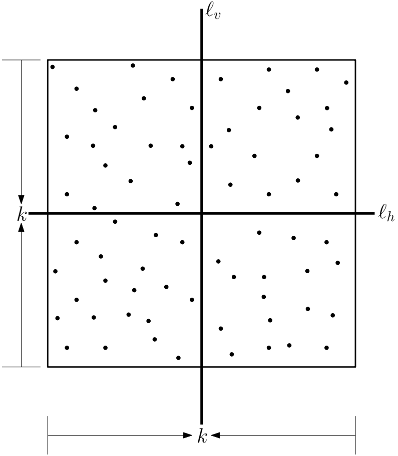


Figure: partition χ into four sub-squares, each of size $\frac{k}{2} \times \frac{k}{2}$, using a horizontal line ℓ_h and a vertical lines ℓ_v.

Approximation scheme for DdIS problem

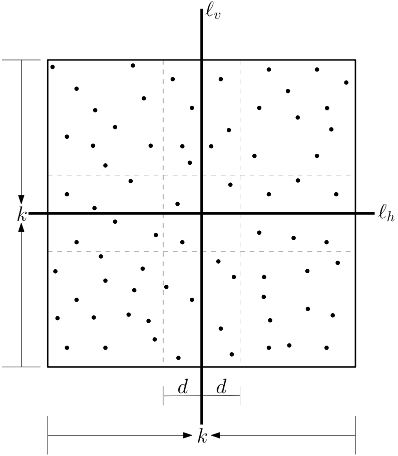


Figure: $\mathcal{Q}_1 \subseteq \mathcal{Q}$, the subset of points in χ which are at most d distance away from ℓ_h and/or ℓ_v

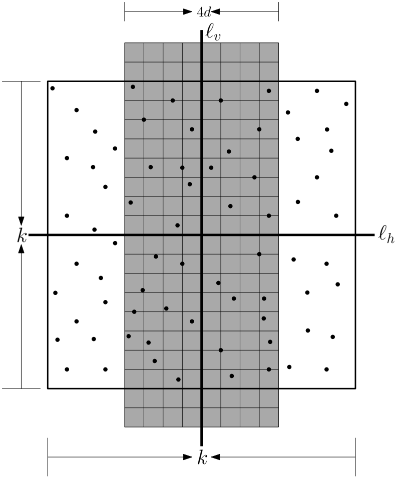
Approximation scheme for DdIS problem

Let $\mathcal{Q}_2$ be a maximum cardinality subset of $\mathcal{Q}_1$ such that all the points in $\mathcal{Q}_2$ are pair wise distance-d independent in p.

Lemma

$|\mathcal{Q}_2| \leq O(k)$.

Approximation scheme for DdIS problem



Approximation scheme for DdIS problem

Lemma

The solution produced for the cell χ (of size $k \times k$) in the aforesaid process is optimum, and the time complexity of the proposed algorithm is $k^2 m^{O(k)}$, where $m = |\mathcal{Q}|$.

$T(m,k) = 4 \times T(m, \frac{k}{2}) \times m^{O(k)} + O(k^2)$, **which is** $k^2 \times m^{O(k)}$ **in the worst case.**

Theorem

Given a set P of n points (centers of the unit disks) in the plane and an integer $k > 1$, the proposed scheme produces a DdIS of size at least $\frac{1}{(1+\frac{1}{k})^2}|OPT|$ in $k^2 n^{O(k)}$ time, where $|OPT|$ is the cardinality of a GMDdIS.

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THANK YOU